QUANTUM CORRELATIONS: PRESERVATION AND APPLICATIONS

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SUCHETANA GOSWAMI

DEPARTMENT OF PHYSICS
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- *Protecting quantum correlations in presence of generalised amplitude damping channel: the two-qubit case*,
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- *Preservation of a lower bound of quantum secret key rate in the presence of decoherence*

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- *Operational nonclassicality of local multipartite correlations in the limited-dimensional simulation scenario*

  C. Jebaratnam, Debarshi Das, **Suchetana Goswami**, Radhakrishna Srikanth, A. S. Majumdar, *J. Phys. A: Math. Theor.* **51**, 365304 (2018).

- *Bipartite qutrit local realist inequalities and the robustness of their quantum mechanical violation*

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Dedicated to my father

Contents

Chapter 1

Introduction

Quantum information theory has opened up new directions in the scientific community, combining the basics of information processing and the fundamental concepts of quantum physics. It provides us with different encouraging results like quantum teleportation [1], super dense coding [2], device independent certifications of cryptographic protocols [3–5] and many more. The subject has developed for decades giving the detailed ideas of the resources and directions to perform such modern communication based jobs. Quantum correlations are one of the most important resources for performing different information processing tasks.

In the year 1935, a completely new perspective towards quantum mechanics was introduced by A. Einstein, B. Podolsky and N. Rosen (EPR) through their article "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [6]. This new aspect shook up the physics community and opened up an extensive area of research. It gives the non-classical manifestation of physical reality in terms of quantum entanglement. The term "entanglement" was coined by E. Schrödinger while he was re-explaining the theory proposed by EPR [7]. It represents the existence of a global state of a composite (multi-partite) system which can not be expressed as a convex mixture of product states of the individual subsystems. EPR concluded their article showing that the quantum description of physical reality is not complete and further posed the question about the existence of such complete theory. Agreeing with the conclusion of EPR, J. Bell constructed the local hidden variable (LHV) model to complete the expression for describing the physical reality with the help of quantum formalism [8]. Here, to come up with the LHV model, J. Bell made 3 assumptions, which are,

1. the condition of realism, i.e. the result of any measurement on a particle is solely determined by the intrinsic properties of the particle,

2. the condition of locality, i.e. the result obtained at one point, does not depend on the action taken place at the spatially separated point,

3. free will, i.e. the local apparatus setting does not depend on the hidden variables which determine the local results.

Based on these assumptions, J. Bell derived an inequality, violation of which corresponds to the physical reality of nature that can be explained by quantum mechanics. The third assumption of free will is valid for any physical process. So, a theory that reproduces the quantum mechanics can be constructed when any one or both the first two assumptions is not valid.

1.1 Quantum correlations

In quantum information theory, there exists mainly three types of non-local correlations for multi-partite systems, namely, quantum entanglement [6, 9], quantum steering [7, 10, 11] and Bell-nonlocality [8, 12]. The paradox posed by EPR demonstrated for the first time, the possibility of creating non-classical and non-local correlations with the help of entanglement, which Schrödinger tried to re-visit in terms of quantum steering for pure quantum states [7]. Subsequently, the pioneering work of Bell [8] paved the way for mathematically distinguishing quantum correlations from those arising through a local realist description of physical phenomena. More recently, it has been realized that quantum correlations could be classified into hierarchical categories [10, 11] with entanglement being the weakest, followed by steering and Bell-nonlocality respectively. Also, quantum steerability for mixed state has been defined the same time [10, 11] based on the local hidden state (LHS) model proposed by E. Schrödinger earlier.

In this thesis, we are mainly interested in two types of quantum correlations, namely, quantum entanglement and quantum steering. The features of quantum entanglement are based on the inseparability of joint state of the composite quantum system and it plays the role of the basic building block of quantum information theory. On the other hand, in case of quantum steering, the shared quantum state of the composite system is considered to be an entangled state a prior. More precisely, the concept of steering is ingrained in the fact that if the bipartite state (shared between Alice (A) and Bob (B)) is said to be steerable from Alice to Bob then Alice can convince Bob by some local operation and classical communication that the state they are sharing is entangled (while Bob does not trust Alice). Basically, a bipartite quantum state exhibits steering if the assemblage i.e. the set of the conditional states prepared on one side (un-trusted party's side) by performing local measurements on the other side cannot be modeled by a description known as the local hidden state (LHS) model (i.e., Bob holds a grand ensemble whose elements are classically correlated with the outcomes of Alice's measurement).

In the present thesis work, we deal with the technique of preservation of these quantum correlations from environmental noise and identification of them. The non-local and inseparable features of quantum correlations enable us to perform various information processing tasks, such as super dense coding [2], quantum teleportation [1], quantum error correction [13], 1-sided and both-side device-independent quantum key distribution [3–5], quantum computing [14] and many more. For the practical implementation of these tasks using the quantum correlations as useful resources, the concerned system has to interact with the noisy environment. The quantum correlations are usually fragile to these noisy environment. So it is one of the most important jobs in any information processing task to preserve the quantum correlation in presence of noise, at least by some amount. This is the general philosophy for controlling decoherence of a quantum system.

On the other hand, to use these correlations as resources, it is important to know from which particular state the correlation has been generated. The procedure of uniquely identifying a particular unknown quantum state and the corresponding measurement settings, while we have a known quantum correlation (a set of probability distributions) in hand, is called the process of self-testing of the given quantum state. Another important problem in quantum information theory is the separability problem, which deals with the identification of any unknown quantum state. In quantum information theory, the main way of identifying entanglement in a given bipartite state is through the separability criterion. Also, there are methods based on direct measurement of observables (which are single setting measurements) such as entanglement witnessing [15] which have been experimentally realized.

Recently, quantum coherence [16] has come to be appreciated as one of the fundamental features of quantum information theory. It has been realized that coherence embodies basic "quantumness" responsible for superposition of quantum states, from which all quantum correlations arise in composite systems. So, the relation of coherence with other resources in quantum theory and its use as an effective resource form an interesting arena of study.

1.2 Outline of the thesis

In this thesis, quantum correlations that are taken in consideration, are entanglement and steering. It has already been mentioned that the interaction of any quantum system with the environment weakens any kind of quantum correlations in general. Here, we consider an environmental interaction governed by the generalized amplitude damping channel (GADC) [17] and likewise both entanglement and steering show a fragile nature when the system has been allowed to interact with the channel. Our motivation is to protect these correlations from the noise atleast by some amount. First we employ the technique of weak measurement [18] which includes a weak measurement followed by the environmental interaction and a reverse weak measurement. But this technique does not help to preserve the correlations for a broad range of the state parameter or channel parameters as it does in the case of Amplitude damping channel (ADC). So, we propose a new and more general technique for the purpose of preservation. Here, we first obtain the unitary dilation corresponding to the Kraus operators of the particular channel in consideration. Then we obtain the inverse of the unitary dilation which basically gives the inverse action of the original channel action. From this inverse unitary matrix, the set of Kraus operators can be obtained, corresponding to the inverse map. These Kraus operators can be considered as the elements of a positive operator valued measure (POVM) and by performing selective measurements on the quantum state, we show that it is possible to preserve the quantum correlations when the state interacts with the noisy environment through GADC. Note that, the unitary dilation obtained corresponding to the channel action is not unique, so there is a scope to find different sets of POVM for the preservation of the correlations and to choose the most suitable element of POVM for a particular case. This work is discussed in Chapter (3).

In the next part of the thesis, we deal with the application based approach of quantum correlations. This includes the self-testing of quantum steering and the detection of quantum entanglement in unknown two-qubit states and the study of the interplay of quantum entanglement with coherence under some three-qubit quantum operations. Firstly, we consider a steering scenario i.e. basically a 1-sided device independent (1SDI) situation. In this scenario, we propose a protocol to self-test any pure bipartite entangled state and the corresponding measurement settings. For this purpose, we consider two steering inequalities, fine-grained uncertainty based inequality (FGI) [19] and analog of the ClauserHorneShimonyHolt inequality (ACHSH) [20] for steering. Violation of FGI gives the idea of the fact that if the quantum state shared is a pure or a mixed entangled state. It has been shown that the maximal violation of FGI is obtained only when the shared quantum state is a pure entangled state. After identifying the class of the state and the corresponding measurement settings with FGI, we either employ the ACHSH inequality or a quantity called mutual predictability to exactly pinpoint the particular state shared, upto some local unitaries. This problem is discussed in Chapter (4).

In Chapter (5), we propose a technique to uniquely identify if an arbitrary bipartite quantum state is entangled or not. For the purpose of detecting entanglement in an unknown quantum state, we use a determinant based separability criterion and by finding the weak values [18,21] corresponding to a suitably chosen Hamiltonian, with the given unknown quantum state being pre-selected and the post-selective measurement is done in the computational basis, in the procedure of weak measurement. This weak measurement involves the application of a global unitary which can be shown to be decomposed in local ones, when we restrict ourselves within pure states. The process is also shown to be robust under the errors arising from the inappropriate choice of the weak interaction Hamiltonian. In this protocol, to find if the state is entangled or not, it is sufficient to have two copies of the given state at a time, which shows a clear advantage over the previously proposed similar protocols where the resource requirement was more. Also, the protocol can be experimentally implemented through a circuit given in Chapter (5).

From another application based point of view, we deal with the interplay of quantum correlation and quantum coherence under few three-qubit cloning operations. On the basis of the definition of coherent and incoherent operations, we separate these cloning operations. It has been realized that coherence embodies basic "quantumness" responsible for the superposition of quantum states, from which all quantum correlations arise in composite systems. The relation of coherence with other resources in quantum theory forms an interesting arena of study. In a recent work, Streltsov et al. [22] have provided an important insight into the linkage of coherence with entanglement. Based upon the observation that two-qubit incoherent operations can generate entanglement only when the input state is coherent, they have shown that the input state coherence provides an upper bound on the generated two-qubit entanglement. The connection between coherence and nonlocal resources such as entanglement is important to understand from both the perspective of quantum foundations and information theoretic applications. This problem has been discussed in Chapter (6).

To deal with all these problems, we first go through few definitions and concepts that are to be taken into consideration throughout the whole thesis. These are discussed in Chapter (2). Finally at the end, some future directions and conclusions have been discussed in Chapter (7).

Chapter 2

Backdrop of the thesis

The whole thesis work mainly revolves around the preservation and applications of quantum correlations. As mentioned in Chapter (1), we consider two types of non-local quantum correlations namely, quantum entanglement and steering. We study the influence of the GADC on these correlations and try to prescribe some definite method of preservation of them under the decoherence. On the other hand, different protocols dealing with identification of the quantum states are described in steering scenario or for entangled system. Also, we show an inter-dependency of quantum coherence with entanglement as another application based approach. In this chapter we introduce a few basic features of quantum foundation theory, which are useful for all the chapters of the thesis.

2.1 Quantum entanglement

For any Hilbert spaces $\mathcal{H}$, let the space of all linear operators be denoted by $\mathcal{L}(\mathcal{H})$, and the set of all density matrices be $\mathcal{P}_+(\mathcal{H})$. Now, in case of a bipartite system, consider two parties, Alice and Bob, each separately possessing a two-level quantum system (qubit) with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Also, consider the pointer system of the measuring apparatus to be a quantum system with Hilbert space $\mathcal{H}_E$. Now, any bipartite quantum state, that can be written as a convex mixture of product states is called a *separable* state i.e.

$$\rho_{sep} = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{2.1}$$

for any $\rho_i^A \in \mathcal{P}_+(\mathcal{H}_A)$, $\rho_i^B \in \mathcal{P}_+(\mathcal{H}_B)$ and probability distribution $\{p_i\}_i$. Otherwise the state of the composite system is said to be an entangled state.

In quantum information theory, there are various ways to detect entanglement in a given biparite state, along with its quantification [15], namely concurrence [23, 24], entanglement of formation [25], geometric measure of entanglement [26], entanglement cost and entanglement of distillation [27], negativity [28, 29] and many more. Concurrence is one of the most important measures of entanglement and for a bipartite state ρ_{AB} it gives a sufficient quantification in this regard and it is defined as

$$C(\rho_{AB}) = max\{(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), 0\} \tag{2.2}$$

with, λ_1, λ_2, λ_3, λ_4 being the eigenvalues of the matrix ρ_f in the descending order, with $\rho_f = \rho_{AB}.\tilde{\rho}_{AB}$ ($\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y).\rho_{AB}.(\sigma_y \otimes \sigma_y)$, σ_y being the Pauli Y-matrix). Maximally entangled states have concurrence 1. In this thesis, as we restrict ourselves to bipartite quantum systems, we consider concurrence as a measure of entanglement.

2.2 Quantum steering

Now let us consider the following scenario which gives the idea about quantum steering as another non-local property of a given bipartite quantum system. Two spatially separated parties, say Alice (A) and Bob (B), share an unknown quantum system $\rho_{AB} \in \mathcal{P}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$ with the Hilbert space dimension of Alice's subsystem being arbitrary (uncharacterized) and the Hilbert space dimension of Bob's subsystem being fixed. Alice performs a set of uncharacterized measurements (i. e., Alice's measurement operators $\{M_{a|x}\}_{a,x}$ are elements of unknown POVM; $M_{a|x} \geq 0 \ \forall a, x$; and $\sum_a M_{a|x} = \mathbb{I}$, $\forall \ x$) on her part of the shared bipartite system ρ_{AB} to prepare the set of conditional states on Bob's side. Here $x = 0, 1, 2, ...$ denotes Alice's choice of measurement settings and $a = 0, 1, 2, ...$ denote the outcomes of Alice's measurement. Such a steering scenario is called 1-sided device-independent (1SDI) since Alice's measurements are treated as black-box measurements. This scenario is schematically demonstrated in Fig. (2.1). The steering scenario is characterized by an assemblage $\{\sigma_{a|x}\}_{a,x}$ [30] which is the set of unnormalized conditional states on Bob's side. Each element in the assemblage is given by $\sigma_{a|x} = p(a|x)\rho_{a|x}$, where $p(a|x)$ is the conditional probability of getting the outcome a when Alice performs the measurement A_x; $\rho_{a|x}$ is the normalized conditional state on Bob's side. Quantum theory predicts that all valid assemblages should satisfy the following criteria:

$$\sigma_{a|x} = Tr_A[(M_{a|x} \otimes \mathbb{I})\rho_{AB}] \quad \forall \sigma_{a|x} \in \{\sigma_{a|x}\}_{a,x}. \tag{2.3}$$

In the above scenario, Alice demonstrates steerability to Bob if the assemblage does not have a local hidden state (LHS) model, i.e., if for all a, x, there is no decomposition of $\sigma_{a|x}$ in the form,

$$\sigma_{a|x} = \sum_{\lambda} p(\lambda)p(a|x,\lambda)\rho_{\lambda}, \tag{2.4}$$

where λ denotes classical random variable which occurs with probability $p(\lambda)$; ρ_λ are called local hidden states which satisfy $\rho_\lambda \geq 0$ and $Tr(\rho_\lambda) = 1$.

We now consider a steering scenario in which the trusted party, Bob, performs a set of POVMs with elements $\{M_{b|y}\}_{b,y}$ ($M_{b|y} \geq 0 \; \forall b, y$; and $\sum_b M_{b|y} = \mathbb{I} \; \forall y$) on the conditional states prepared by Alice's unknown POVMs turning the assemblage $\{\sigma_{a|x}\}_{a,x}$ into measurement correlations $p(ab|xy)$, where $p(ab|xy) = Tr(M_{b|y}\sigma_{a|x})$. Here $y = 0, 1, 2, ...$ denotes Bob's choice of measurement setting and $b = 0, 1, 2, ...$ denotes outcome of Bob's measurement. The correlation $p(ab|xy)$ detects steerability from Alice to Bob, iff it does not have a decomposition as follows [10, 11]:

$$p(ab|xy) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)p(b|y,\rho_\lambda) \quad \forall a, x, b, y; \tag{2.5}$$

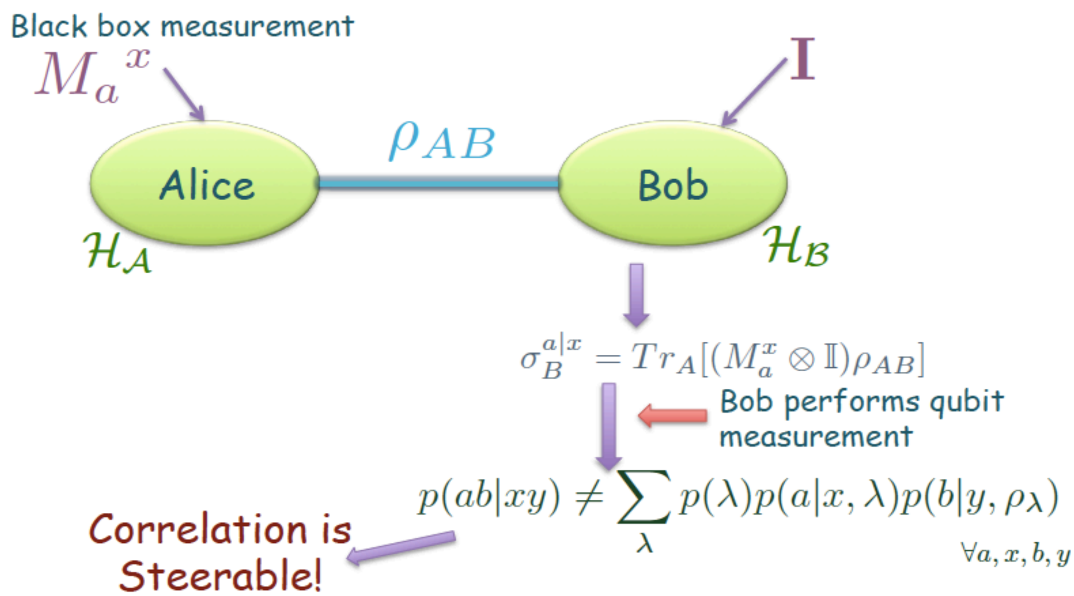

Figure 2.1: Schematic diagram for quantum steering for a bipartite system.

where, $\sum_\lambda p(\lambda) = 1$, and $p(a|x, \lambda)$ denotes an arbitrary probability distribution arising from local hidden variable (LHV) λ (λ occurs with probability $p(\lambda)$) and $p(b|y, \rho_\lambda)$ denotes the quantum probability of outcome b when measurement B_y is performed on local hidden state (LHS) ρ_λ. Hence, the box $p(ab|xy)$ will be called steerable correlation iff it does not have a LHV-LHS model. Steerable correlations are certified through the violation of a steering inequality [31].

Similar as quantum entanglement, there are different quantifiers that certifies quantum steering in a bipartite system. One of them is steerable weight [32] which is a convex steering monotone [33]. Consider the following decomposition of an arbitrary assemblage $\{\sigma_{a|x}\}_{a,x}$:

$$\sigma_{a|x} = p_s \sigma^S_{a|x} + (1 - p_s)\sigma^{US}_{a|x} \quad \forall a, x, \tag{2.6}$$

where $0 \le p_s \le 1$, $\sigma^S_{a|x}$ is a steerable assemblage and $\sigma^{US}_{a|x}$ is an element of unsteerable assemblage having LHS model. The weight of the steerable part p_s minimized over all possible decompositions of the given assemblage $\{\sigma_{a|x}\}_{a,x}$ gives the steerable weight $SW(\{\sigma_{a|x}\}_{a,x})$ of that assemblage.

Also, we consider another measure for steering based on the violation of a prescribed steering inequality, in which A performs two black-box (as Alice's side is considered to be uncharacterised) dichotomic measurements A_0 and A_1 and B performs two qubit-measurements in mutually unbiased bases, given by B_0 $(= \sigma_z)$ and B_1 $(= \sigma_x)$. Then the necessary and sufficient condition for quantum steering as introduced by Cavalcanti-Foster-Fuwa-Wiseman (CFFW) [20] can be written in terms of an inequality (also termed as Analog CHSH (ACHSH) inequality for steering) as stated below.

$$\begin{aligned} &\sqrt{\langle (A_0 + A_1)B_0\rangle^2 + \langle (A_0 + A_1)B_1\rangle^2} \\ +&\sqrt{\langle (A_0 - A_1)B_0\rangle^2 + \langle (A_0 - A_1)B_1\rangle^2} \leq 2, \end{aligned} \tag{2.7}$$

with, $\langle A_x B_y\rangle = \sum_{a,b}(-1)^{a\oplus b}p(ab|xy)$. Violation of this inequality quatifies steerability of an entangled state.

Other than the inequality given in Eq. (2.7), there are different inequalities and quantifiers to certify steering. One of them i sthe fine-grained uncertainty based steering inequality (FGI) [19]. Further we show that the maximal violation of this particular inequality certifies pure entangled states corresponding to the suitable choice of measurement settings. The FGI for steering is given as follows,

$$P(b_{B_0} \mid a_{A_0}) + P(b_{B_1} \mid a_{A_1}) \leq 1 + \frac{1}{\sqrt{2}}. \tag{2.8}$$

All the quantities mentioned in this chapter, giving the measures of the quantum correlations under consideration, will be used throughout the whole thesis.

2.3 Quantum coherence

Coherence has a long-standing history in the fields of classical and quantum optics. The accuracy of any optical technology mainly rely on the coherence of the particles in the input beam. But, quantum coherence is not restricted in the field of optics any more. On the contrary, this plays a crucial role being the resource for many information processing tasks in quantum technology. Similar to quantum entanglement, hence this also has a well-defined resource theory, where the incoherent operations are considered as the free operations and the incoherent states are the free states [16]. Incoherent operations are those which can not generate coherence starting from an incoherent state. There are different classes of incoherent operations, such as maximally incoherent operation (MIO), strictly incoherent operation (SIO), physical incoherent operation (PIO) and incoherent operations (IO), making an hierarchical situation as [34–36],

$$PIO \subset SIO \subset IO \subset MIO \tag{2.9}$$

On the other hand, there are different measures to quantify quantum coherence as similar to that of entanglement. They are, the distance based measure of coherence which is defined in terms of the distance between the particular coherent state and the nearest incoherent state, l_1 norm of coherence which is calculable from the density matrix structure of the corresponding state, entropic measure of coherence etc. In general, coherence is a basis dependent quantity and also it might exist in a single-partite systems. Based on the superposition principle, an arbitrary state can be classified into two types: incoherent and coherent state. A state ρ is said to be incoherent if it can be expressed in the form

$$\rho = \sum_i \rho_i \left|i\right\rangle \left\langle i\right| \tag{2.10}$$

where $|i\rangle$ represents a fixed reference basis of the state. Otherwise, it is said to be a coherent state. This definition holds not only for single qubit systems but also for higher dimensional quantum systems. Now, an incoherent quantum operation is defined as a completely positive trace preserving map which takes an incoherent state into another. There are different classes of incoherent operation as well [16]. Mathematically, an incoherent quantum operation Λ can be written as

$$\Lambda(\rho) = \sum_l K_l \rho K_l^\dagger \tag{2.11}$$

where the operators K_l are incoherent Kraus operators. In this thesis, for the analysis of coherence, we will employ the l_1 norm measure [150] defined as

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \tag{2.12}$$

Other relevant features related to this type of quantum resource, required for the study of the thesis interest are discussed in details in Chapter (6).

2.4 Quantum cloning

In the year 1982, Wootters and Zurek proposed the famous *no cloning* theorem which makes a distinct classification between the quantum and classical information theory [37]. They have shown that it is not possible to copy any unknown quantum state with maximum probability. Hence, based on this there are different security and cryptographic protocols have been introduced in the literature of quantum theory. But following the result of [37], there have been a numerous number of protocols have been posed representing different types of quantum cloning, where none shows a power to copy the state exactly but with some given probability. They are, *Wootter-Zurek* cloning machine [37], *Buzek-Hillary* cloning machine [38], *phase-covariant* cloning machine [39], *state-dependent universal* cloning machine [40], *asymmetric* cloning machine [41] etc.

Here let us now consider a few mathematical transformations which govern the cloning operations in a given scenario. Let us first consider the three qubit quantum operation given by [38] the following equations and termed as BH cloning transformation.

$$\begin{aligned} |0\rangle_a |0\rangle_b |0\rangle_c \;\; \rightarrow \;\; & \sqrt{\frac{2}{3}} |0\rangle_a |0\rangle_b |0\rangle_c + \\ & \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) |1\rangle_c \end{aligned} \tag{2.13}$$

$$
\begin{aligned}
|1\rangle_a |0\rangle_b |0\rangle_c \;\; \rightarrow \;\; & \sqrt{\frac{2}{3}} |1\rangle_a |1\rangle_b |1\rangle_c + \\
& \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) |0\rangle_c
\end{aligned} \tag{2.14}
$$

The above cloning operation can be considered as an input state independent cloning transformation. Whereas there are cloning machines whose operation depends on the state in the input end. In this type of cloning [40], the cloner operates a unitary operation on the composite Hilbert space of the three party state in the following way,

$$
\begin{aligned}
|0\rangle |0\rangle |X\rangle \;\; \rightarrow \;\; & a |00\rangle |A\rangle + b_1 |01\rangle |B_1\rangle + b_2 |10\rangle |B_2\rangle + \\
& c |11\rangle |C\rangle
\end{aligned} \tag{2.15}
$$

$$
\begin{aligned}
|1\rangle |0\rangle |X\rangle \;\; \rightarrow \;\; & \tilde{a} |11\rangle \left|\tilde{A}\right\rangle + \tilde{b_1} |01\rangle \left|\tilde{B_1}\right\rangle + \tilde{b_2} |10\rangle \left|\tilde{B_2}\right\rangle + \\
& \tilde{c} |00\rangle \left|\tilde{C}\right\rangle
\end{aligned} \tag{2.16}
$$

Here the state $|X\rangle$ represents the initial ancillary machine state and $|A\rangle$, $|B_1\rangle$, $|B_2\rangle$, $|C\rangle$, $\left|\tilde{A}\right\rangle$, $\left|\tilde{B_1}\right\rangle$, $\left|\tilde{B_2}\right\rangle$, $\left|\tilde{C}\right\rangle$ signify the ancillary machine state at the output end. As the operation of cloning is unitary, the coefficients in each case should satisfy the normalization conditions,

$$
a^2 + {b_1}^2 + {b_2}^2 + c^2 = 1 \tag{2.17}
$$

$$
\tilde{a}^2 + \tilde{b_1}^2 + \tilde{b_2}^2 + \tilde{c}^2 = 1 \tag{2.18}
$$

In this thesis, we consider a few of these cloning machines (including the cloning machines discussed in this section) in 3-qubit form, where the first qubit is the state that is to be copies, the second one is the store-qubit and the third one plays the role of an ancillary system. We categorize these cloning operations in terms of coherent or incoherent operations and study the nature of the correlation generated at the output end when the ancillary part is traced out. The other necessary details of the cloning machines taken into consideration are discussed in Chapter (6).

All the other tools used for the purpose of the thesis, are discussed in details in the corresponding chapter.

Chapter 3

Protecting quantum correlations in presence of generalized amplitude damping channel for pure two-qubit entangled states

Any kind of quantum resource useful in different information processing tasks is vulnerable to several types of environmental noise. Here we study the behaviour of quantum correlations such as entanglement and steering in two-qubit systems under the application of the generalized amplitude damping channel and propose some protocols towards preserving them under this type of noise. First, we employ the technique of weak measurement and reversal for the purpose of preservation of correlations. We then show how the evolution under the channel action can be seen as an unitary process. We use the technique of weak measurement and most general form of selective positive operator valued measure (POVM) to achieve preservation of correlations for a significantly large range of parameter values.

3.1 Introduction and motivation

Non-local features of quantum correlations enable us to perform various information processing tasks, as discussed in Chapter (1). While practically implementing such tasks, there is always an interaction taking place between the concerned quantum system and the noisy environment, due to which the useful resources governed by the quantum correlations get diminished in most of the cases. So it is one of the most important jobs in any information processing task to preserve the quantum correlation in presence of noisy environment, at least by some amount.

There are various well-known forms of decohence, modelled by, the depolarising channel, dephasing channel, amplitude damping channel (ADC), generalized amplitude damping channel (GADC), and so on [17, 42–45]. It has been shown that it can never be possible to enhance quantum correlation in two-qubit system by unital operations, whereas for some initial states it might be possible to enhance or generate quantum correlation when the interaction is taking place through some non-unital channel [46]. For example, interaction through amplitude damping channel can enhance the teleportation fidelity for a particular class of two-qubit entangled state [47, 48]. On the other hand, using the technique of weak measurement, one can improve the fidelity of teleportation [49] as well as the secret key rate for one-sided device independent key distribution (1-SDIKD) [50], while the interaction is taking place through the ADC.

There exist other ways to protect quantum correlations from environmental noise, such as by employing quantum Zeno effect [51, 52], frequent unitary interruptions (bang-bang pulse) [53–56], strong continuous coupling [14, 57, 58], etc. In the present work, we focus on the problem of preservation of quantum correlations under the decoherence arising from the action of the generalized amplitude damping channel. Note that all of the aforesaid decoherence control processes are dynamical in nature: one needs to follow the dynamics of the system in order to implement each such control process. On the other hand, the environment action, considered in the present work is of static nature and hence the aforesaid decoherence controlling mechanism will not work, in general, for our case.

Here, we confine our studies within quantum entanglement and quantum steering. Each of these two correlations decreases under the action of generalized amplitude damping channel which is a non-unital channel. Now, for the purpose of preservation of non-local correlation, we start with pure (maximally and non-maximally) entangled states, and first employ the technique of weak measurement which has been used in case of the standard ADC [50]. In the case of weak measurement, as first proposed in [18], the interaction between the system and the apparatus is taken to be very weak, along with two measurements termed as preselection and post-selection [21]. To use weak measurement as a procedure for preservation of correlations in a quantum state, one has to do the weak measurement, and the reverse weak measurement to be followed at the end of the protocol. The procedure of weak measurement has been used in a huge number of protocols to study different interesting phenomena in quantum theory, such as spin Hall effect [59], wave particle duality using cavity-QED experiments [60], superluminal propagation of light [61, 62], direct measurement of the quantum wave function [63], measurement of ultrasmall time delays of light [64], observing Bohmian trajectories of photons [65, 66] and also for detection of entanglement with minimal resources [67].

Next, to make the technique of preservation of non-local correlations more general for environmental noise, we find the unitary dialation corresponding to the completely positive trace-preserving evolution of the GADC, starting from its known Kraus representation [68]. After finding the inverse of this unitary, we construct the most general form of operator-sum representation (Kraus representation) of an approximate inverse map, which is not unique. Employing these Kraus operators individually as the elements of a POVM, we show that it is possible to preserve the correlations of the initial state up to certain extents for a broad range of state parameter and damping coefficients. It should be noted that in all the cases one has to employ selective POVM, as non-selective POVM corresponds to an unitary evolution, and under local unitary it is not possible to generate or enhance any kind of quantum correlation [69]. Although our method (to be described in Sec. (3.4) below) may appear to be quite specific towards tackling the noise of GADC, nevertheless, as a method, it has a general appeal in the sense that it can, in principle, be applicable to any noise model - provided we have the prior information about the noise model.

3.2 Preliminaries

In this work, we study the behaviour of the aforesaid two quantum correlations (entanglement and steering) under the application of environmental noise expressed in the form of generalized amplitude damping. GADC can be obtained by solving the optical master equation in presence of a squeezed thermal bath and this channel describes the effect of environmental dissipation in a finite temperature bath. The Kraus operators of the corresponding channel are given by,

$$\begin{aligned} K_1 &= \sqrt{\nu}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{bmatrix} \\ K_2 &= \sqrt{\nu}\begin{bmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{bmatrix} \\ K_3 &= \sqrt{1-\nu}\begin{bmatrix} \sqrt{\eta} & 0 \\ 0 & 1 \end{bmatrix} \\ K_4 &= \sqrt{1-\nu}\begin{bmatrix} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{bmatrix} \end{aligned} \tag{3.1}$$

where, $\nu \in [0,1]$ reflects the temperature of the bath and $\eta \in [0,1]$ is the parameter representing the rate of dissipation due to the bath action. Note that, for $\nu = 1$, the Kraus representation in Eq. (3.1) reduces to that of an ADC, for which the environment is assumed to be at zero temperature. The GADC operation Λ is assumed to be acting here on one of the qubits of a two-qubit state

$$\rho_{AB} : \Lambda(\rho_{AB}) = \sum_{i=1}^{4} (\mathbb{I}_A \otimes K_i)\rho_{AB}(\mathbb{I}_A \otimes K_i^{\dagger}) \tag{3.2}$$

In previous works, it has been shown that, it is possible to subdue the effect of environmental interaction through ADC by the application of the technique of weak measurement and its reversal [49, 50]. In the similar way, in this work, we first study the effect of weak measurement technique when the environmental interaction is taking place through GADC. Here, before the environmental interaction takes place with the particle in consideration (say, B) of the bipartite system AB, weak measurement (WM) with a strength w is performed on the same. Basically in this case, the detector detects the system with probability w if and only if the state of B is in $|1\rangle$ ($= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$) and hence the measurement operator W_1 corresponding to this scenario is given as,

$$W_1 = \sqrt{w}\,|1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{w} \end{bmatrix} \tag{3.3}$$

Note that, the matrix in Eq. (3.3) is singular. Hence this is not effective to implement for the reverse weak measurement. So, for the purpose of weak measurement, we consider the measurement operator corresponding to the scenario when the system is not detected by the measuring apparatus. The measurement operator W_0 corresponding to this situation can be evaluated by using the relation, $W_1^\dagger W_1 + W_0^\dagger W_0 = \mathbb{I}$. Hence,

$$W_0 = |0\rangle\langle 0| + \sqrt{1-w}\,|1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-w} \end{bmatrix} \tag{3.4}$$

As the matrix in Eq. (3.4) is a reversible one, application of the inverse of it leads back the system to its original state. According to the our protocol, after the weak measurement is done, the particle in consideration interacts with the environment through the GADC and lastly reverse weak measurement (RWM) is done on the same. The Kraus operator corresponding to the reverse weak measurement is given below.

$$R_0 = \begin{bmatrix} \sqrt{1-r} & 0 \\ 0 & 1 \end{bmatrix} \tag{3.5}$$

where, r is the strength of the reverse weak measurement (we consider it different from the weak measurement strength w to make sure that there is a freedom of choice for different efficiencies of weak and reverse weak measurement).

After the implementation of the technique of weak measurement, now we propose another, our more general approach for the purpose of preservation of steering and entanglement of the bipartite state while interacting with environment through GADC. Here we consider the unitary dilation corresponding to the completely positive trace preserving (CPTP) map governed by the GADC. The unitary dilation corresponding to the Kraus operator representation given in Eq.(3.1), is not unique and can be obtained considering a two-qubit ancilla for the action of GADC on each side of the two-qubit system. The action of this unitary (say, U_{SB}) on the initial state of system (B) plus ancilla (S) gives the state after the environmental interaction taken place through GADC. In the next step, we find the inverse of this unitary (U_{SB}^{-1}) from which one can find the corresponding Kraus operator representation of the evolution. These individual Kraus operators can be considered as elements of the most general POVM. Note here that the reduced CPTP map (acting on B), formed out of U_{SB}^{-1}, is not necessarily the inverse of the given GADC (even if such an inverse map exists). This would have been the case, if under a suitable choice of an initial state σ_S of the ancilla S, the reduced state $Tr_S[U_{SB}^{-1}(\sigma_S \otimes \Lambda(\rho_B))U_{SB}]$ of B becomes close to the initial state ρ_B before applying the GADC Λ. Employing the selective POVM, obtained corresponding to the inverse map, either before or after the action of the GADC, we study the concurrence and the steerability (violation of ACHSH inequality) of the final state.

3.3 Employing the technique of weak measurement

As described in the previous section, here we consider that only one side (B) of the bipartite system shared between A and B is interacting with the environment. We study two cases described in the flowchart given below.

1) $\rho_{AB} \xrightarrow{GADC(B)} \rho'_{AB} = \sum_{i=1}^{4}(\mathbb{I}_A \otimes K_i)\rho_{AB}(\mathbb{I}_A \otimes {K_i}^{\dagger})$.

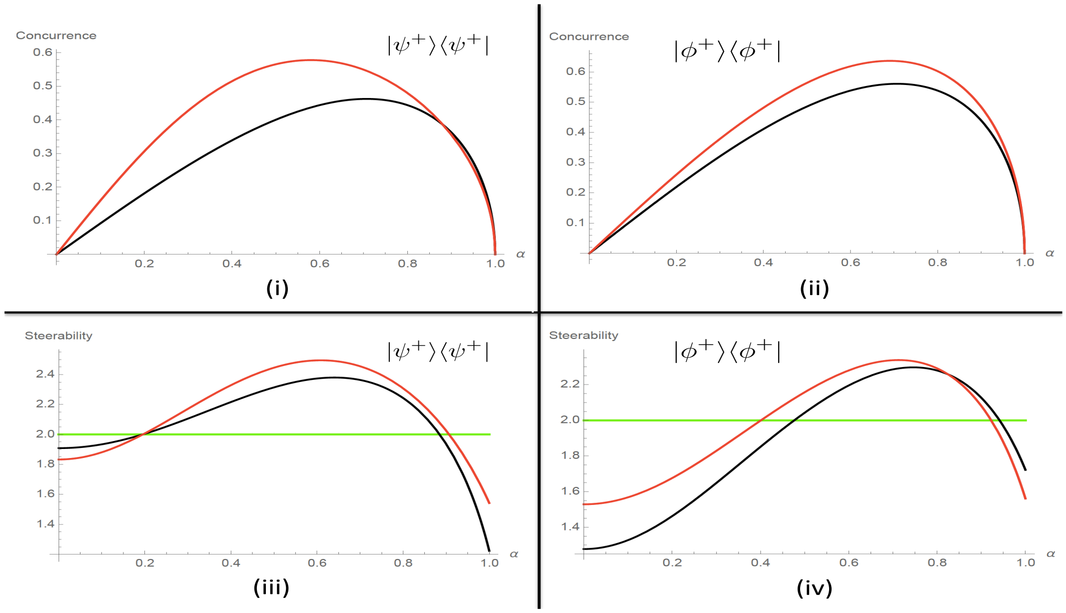

Figure 3.1: (i) Comparison of concurrence while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (ii) Comparison of concurrence while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$, (iii) Comparison of steerability while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (iv) Comparison of steerability while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$. In all the plots, Red curves denote the corresponding function with the weak measurement and Black curves denote the same without weak measurement, where Green lines in the plots (iii) and (iv) denote the limit of the violation of the ACHSH inequality. (For individual plots, ν, η, w and r have been kept fixed.)

2) $$\begin{aligned}\rho_{AB} &\xrightarrow{WM(B)} \rho^w_{AB} = (\mathbb{I}_A \otimes W_0)\rho_{AB}(\mathbb{I}_A \otimes {W_0}^\dagger) \\ &\xrightarrow{GADC(B)} \rho^d_{AB} = \textstyle\sum_{i=1}^4 (\mathbb{I}_A \otimes K_i)\rho^w_{AB}(\mathbb{I}_A \otimes {K_i}^\dagger) \\ &\xrightarrow{RWM(B)} \rho''_{AB} = (\mathbb{I}_A \otimes R_0)\rho^d_{AB}(\mathbb{I}_A \otimes {R_0}^\dagger).\end{aligned}$$

where K_i's, W_0 and R_0 are given in Eq. (3.1), (3.4) and (3.5) respectively. Note that comparison of steerability and concurrence between the states ρ'_{AB} and ρ''_{AB} gives the idea about the fact whether the technique of weak measurement is useful for the preservation of quantum correlation. In the whole chapter, we consider either pure anti-parallel entangled state ($|\psi^\pm\rangle\langle\psi^\pm|$) or pure parallel entangled state ($|\phi^\pm\rangle\langle\phi^\pm|$) in computational basis as the initial state (ρ_{AB}) for all the protocols, where,

$$|\psi^{\pm}\rangle = \alpha\,|01\rangle \pm \beta\,|10\rangle\,, \tag{3.6}$$

$$|\phi^{\pm}\rangle = \alpha\,|00\rangle \pm \beta\,|11\rangle\,. \tag{3.7}$$

In the above cases one must have $\alpha^2 + \beta^2 = 1$, to fulfill the demand of normalization. It is evident from Fig.(3.1), the correlations (entanglement and steering) show a certain amount of improvement for a section of pure states (for some values of state parameter α) under the application of weak measurement technique. Note that, the plots in Fig.(3.1) are corresponding to a particular set of values of the GADC parameters, ν and η. It can be seen that for some other set of values for the channel parameter ν and η, it is possible to preserve both the correlations under the application of weak measurement corresponding to different weak measurement strength w and reverse weak measurement strength r. But the range of channel parameters for this technique showing any improvement is quite small (The plots are shown for a particular set of values of the channel parameters). So in the next section we propose a more general approach for the preservation of quantum correlation using the selective POVM. This approach is general in the sense that it deals with the unitary dilation of the GADC, which is not unique. This gives us the freedom to choose the suitable POVM that preserves the correlation maximally.

3.4 Preserving the correlation considering a unitary dilation of the channel

It is an well-known fact that any quantum channel that corresponds to a physical process can be seen as a CPTP evolution. In this section we try to obtain the unitary dilation of the CPTP evolution corresponding to the GADC and use the same for the purpose of preservation of entanglement and steerability.

3.4.1 To find the unitary dilation and its inverse

Let us start by considering a general CPTP map given as $\mathcal{N} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S)$, with $\mathcal{H}_S$ being the d dimensional Hilbert space associated with a given quantum system S and $\mathcal{B}(\mathcal{H}_s)$ represents the set of all bounded linear operators, $\mathcal{A} : \mathcal{H}_S \to \mathcal{H}_S$. It is obvious that $\mathcal{N}$ has an operator-sum representation (or, Kraus representation) expressed as, $\mathcal{N}(\mathcal{A}) = \sum_{i=1}^{M} L_i \mathcal{A} {L_i}^\dagger$ (M being a finite positive integer) with the Kraus operators L_i satisfying the condition, $\sum_{i=1}^{M} {L_i}^\dagger L_i = \mathbb{I}_d$. Now, let us consider an ancilla system B whose associated Hilbert space $\mathcal{H}_B$ has dimension M. Let, $\{|i\rangle_S\}_{i=1}^{d}$ be an orthonormal basis (ONB) for $\mathcal{H}_S$ while $\{|i\rangle_B\}_{i=1}^{M}$ be an ONB for $\mathcal{H}_B$. Our aim is to find a $dM \times dM$ unitary matrix U_{SB} which corresponds to the map $\mathcal{N}$ such that,

$$L_i = {}_B\langle i| U_{SB} |1\rangle_B \,, \forall i = 1, 2, ..., M. \tag{3.8}$$

Alternatively, for every $\mathcal{A} \in \mathcal{B}(\mathcal{H}_S)$ (thus, $\mathcal{A}$ can also be a density matrix of S), one can write,

$$\mathcal{N}(\mathcal{A}) = Tr_B[U_{SB}(\mathcal{A} \otimes |1\rangle_B \langle 1|){U_{SB}}^\dagger]. \tag{3.9}$$

Note that, for a given CPTP map, its unitary dilation given by the matrix U_{SB} is not unique. The $(\alpha i, \beta 1)$-entry of the matrix can be obtained in the following way,

$$\begin{aligned} u_{\alpha i, \beta 1} &\equiv ({}_S\langle \alpha| \otimes {}_B\langle i|) U_{SB} (|\beta\rangle_S \otimes |1\rangle_B) \\ &= {}_S\langle \alpha| L_i |\beta\rangle_S \,, \ \forall \alpha, \beta = 1, 2, ..., d. \end{aligned} \tag{3.10}$$

For a given set of Kraus operators, L_1, L_2,..., L_M, with the help of Eq. (3.10), it is possible to obtain information about d column vectors of the unitary matrix U_{SB} (with respect to the joint ONB $\{|\alpha\rangle \otimes |i\rangle \, \big| \alpha = 1, 2, ..., d; i = 1, 2, ..., M\}$) and they are given below.

$$\begin{aligned}
\overrightarrow{u_{11}} = (u_{11,11}, \quad & u_{12,11}, ..., u_{1M,11}, u_{21,11}, u_{22,11}, ..., \\
& u_{2M,11}, ..., u_{d1,11}, u_{d2,11}, ..., u_{dM,11})^T, \\
\overrightarrow{u_{21}} = (u_{11,21}, \quad & u_{12,21}, ..., u_{1M,21}, u_{21,21}, u_{22,21}, ..., \\
& u_{2M,21}, ..., u_{d1,21}, u_{d2,21}, ..., u_{dM,21})^T, \\
.. & \\
.. & \\
\overrightarrow{u_{d1}} = (u_{11,d1}, \quad & u_{12,d1}, ..., u_{1M,d1}, u_{21,d1}, u_{22,d1}, ..., \\
& u_{2M,d1}, ..., u_{d1,d1}, u_{d2,d1}, ..., u_{dM,d1})^T
\end{aligned} \tag{3.11}$$

Thus, we obtain an incomplete ONB $\{\overrightarrow{u_{11}}, \overrightarrow{u_{21}}, ..., \overrightarrow{u_{d1}}\}$ of the dM dimensional Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_B$. At this point, we use the method of basis extension to extend this incomplete ONB given in Eq. (3.11) to form the complete ONB $\{\overrightarrow{u_{11}}, \overrightarrow{u_{12}}, ..., \overrightarrow{u_{1M}}, \overrightarrow{u_{21}}, \overrightarrow{u_{22}}, ..., \overrightarrow{u_{2M}}, ..., \overrightarrow{u_{d1}}, \overrightarrow{u_{d2}}, ..., \overrightarrow{u_{dM}}\}$ for $\mathcal{H}_S \otimes \mathcal{H}_B$ and eventually to construct the unitary matrix U_{SB}. The procedure of basis expansion is not unique, but it is restricted by two conditions: i) all the column vectors of U_{SB} should be orthogonal to each other and ii) the individual columns must be normalized. Taking these two constraints into consideration one can construct different forms of U_{SB} starting from Eq. (3.11), all of which represents the same CPTP map $\mathcal{N}$ that we have started with.

Now, let us consider the Kraus representation of GADC (expressed in the computational basis) given in Eq. (3.1) and the evolution of ρ_{AB} is given by, Eq. (3.2). Hence, in this particular scenario of environmental interaction through GADC, we have $M = 4$ and $d = 2$. Thus, in this case, the unitary matrix U_{SB} must be of dimension 8×8. Following the technique mentioned above in Eq.

(3.10), we find only two columns of the 8×8 unitary, which are $\overrightarrow{u_{11}}$ and $\overrightarrow{u_{21}}$,:

$$\begin{aligned}
\overrightarrow{u_{11}} = (u_{11,11} \equiv {}_S\langle 0|\, K_1 \,|0\rangle_S = \sqrt{\nu},\ u_{12,11} \equiv {}_S\langle 0|\, K_2 \,|0\rangle_S = 0,\\
u_{13,11} \equiv {}_S\langle 0|\, K_3 \,|0\rangle_S = \sqrt{(1-\nu)\eta},\ u_{14,11} \equiv {}_S\langle 0|\, K_4 \,|0\rangle_S = 0,\\
u_{21,11} \equiv {}_S\langle 1|\, K_1 \,|0\rangle_S = 0,\ u_{22,11} \equiv {}_S\langle 1|\, K_2 \,|0\rangle_S = 0,\\
u_{23,11} \equiv {}_S\langle 1|\, K_3 \,|0\rangle_S = 0,\ u_{24,11} \equiv {}_S\langle 1|\, K_4 \,|0\rangle_S = \sqrt{(1-\nu)(1-\eta)})^T;
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\overrightarrow{u_{21}} = (u_{11,21} \equiv {}_S\langle 0|\, K_1 \,|1\rangle_S = 0,\ u_{12,21} \equiv {}_S\langle 0|\, K_2 \,|1\rangle_S = \sqrt{\nu(1-\eta)},\\
u_{13,21} \equiv {}_S\langle 0|\, K_3 \,|1\rangle_S = 0,\ u_{14,21} \equiv {}_S\langle 0|\, K_4 \,|1\rangle_S = 0,\\
u_{21,21} \equiv {}_S\langle 1|\, K_1 \,|1\rangle_S = \sqrt{\nu\eta},\ u_{22,21} \equiv {}_S\langle 1|\, K_2 \,|1\rangle_S = 0,\\
u_{23,21} \equiv {}_S\langle 1|\, K_3 \,|1\rangle_S = \sqrt{1-\nu},\ u_{24,21} \equiv {}_S\langle 1|\, K_4 \,|1\rangle_S = 0)^T.
\end{aligned} \tag{3.13}$$

Using the method of basis expansion, and by taking care of the constraints stated previously, we construct several unitary matrices representing the noisy channel (GADC) in consideration. Note that, from Eq. (3.1), for $\nu = \eta = 1$, the channel should represent an identity operation. Keeping all these facts in mind, in this chapter, we concentrate on two separate unitary evolutions corresponding to the GADC which are given in terms of the 8×8 matrix U_{SB} below.

$$U_{SB}^{(1)} =$$

$$\begin{pmatrix}
\sqrt{\nu} & 0 & -\frac{\sqrt{(1-\nu)\eta}}{\sqrt{1-(1-\nu)(1-\eta)}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{\nu(1-\nu)(1-\eta)}}{\sqrt{1-(1-\nu)(1-\eta)}}\\
0 & \sqrt{\eta} & 0 & 0 & \sqrt{\nu(1-\eta)} & 0 & -\sqrt{(1-\nu)(1-\eta)} & 0\\
\sqrt{(1-\nu)\eta} & 0 & \frac{\sqrt{\nu}}{\sqrt{1-(1-\nu)(1-\eta)}} & 0 & 0 & 0 & 0 & -\frac{(1-\nu)\sqrt{\eta(1-\eta)}}{\sqrt{1-(1-\nu)(1-\eta)}}\\
0 & 0 & 0 & \sqrt{\eta(1-\nu)+\nu} & 0 & -\sqrt{(1-\nu)(1-\eta)} & 0 & 0\\
0 & -\sqrt{1-\eta} & 0 & 0 & \sqrt{\nu\eta} & 0 & -\sqrt{(1-\nu)\eta} & 0\\
0 & 0 & 0 & \sqrt{(1-\nu)(1-\eta)} & 0 & \sqrt{\eta(1-\nu)+\nu} & 0 & 0\\
0 & 0 & 0 & 0 & \sqrt{1-\nu} & 0 & \sqrt{\nu} & 0\\
\sqrt{(1-\nu)(1-\eta)} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\eta(1-\nu)+\nu}{\sqrt{1-(1-\nu)(1-\eta)}}
\end{pmatrix} \tag{3.14}$$

$$U_{SB}^{(2)} =$$

$$\begin{pmatrix}
\sqrt{\nu} & 0 & -\frac{\sqrt{\eta(1-\nu)}\sqrt{\nu}}{\sqrt{1-\eta(1-\nu)}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{(1-\eta)(1-\nu)}}{\sqrt{1-\eta(1-\nu)}}\\
0 & \frac{\sqrt{\eta\nu}}{\sqrt{1-(1-\nu)}} & 0 & 0 & \sqrt{(1-\eta)\nu} & 0 & -\sqrt{1-\nu}\sqrt{1-\eta} & 0\\
\sqrt{\eta(1-\nu)} & 0 & \sqrt{1-\eta(1-\nu)} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \sqrt{\eta(1-\nu)+\nu} & 0 & -\sqrt{(1-\eta)(1-\nu)} & 0 & 0\\
0 & -\frac{\sqrt{(1-\eta)\nu}}{\sqrt{1-(1-\nu)}} & 0 & 0 & \sqrt{\eta\nu} & 0 & -\sqrt{1-\nu}\sqrt{\eta} & 0\\
0 & 0 & 0 & \sqrt{(1-\eta)(1-\nu)} & 0 & \sqrt{\eta(1-\nu)+\nu} & 0 & 0\\
0 & 0 & 0 & 0 & \sqrt{1-\nu} & 0 & \sqrt{\nu} & 0\\
\sqrt{(1-\eta)(1-\nu)} & 0 & -\frac{(1-\nu)\sqrt{\eta(1-\eta)}}{\sqrt{1-\eta(1-\nu)}} & 0 & 0 & 0 & 0 & \frac{\sqrt{\nu}}{\sqrt{1-\eta(1-\nu)}}
\end{pmatrix} \tag{3.15}$$

Note that, in these cases the aforesaid GADC appears as a dynamical process in time (as for example, by solving the optical master equation with the initial state of the heat bath taken to be squeezed vacuum). Our aim is to find out the Kraus operator representation of the quantum channel whose unitary dilation corresponds to the inverse of this unitary evolution. As U_{SB} is unitary, we must have $U_{SB}^{-1} = U_{SB}^{\dagger}$. Now under the action of the inverse unitary evolution U_{SB}^{-1}, the state of the system at the output end is given by, $Tr_B[U_{SB}^{\dagger}(\sigma_S \otimes |1\rangle_B \langle 1|)U_{SB}]$, where σ_S is the state of the system the just before the action of the inverse unitary. Now if J_i for $i = 1, 2, 3, 4$ be the Kraus operators corresponding to the channel described by the inverse unitary, one must have,

$$Tr_B[U_{SB}^{\dagger}(\sigma_S \otimes |1\rangle_B \langle 1|)U_{SB}] = \sum_{i=1}^{4} J_i \sigma_S J_i^{\dagger} \tag{3.16}$$

with, $J_i = {}_B\langle i| U_{SB}^{-1} |1\rangle_B$ for $i = 1, 2, 3, 4$. In fact, if $U_{SB} = \sum_{k,l=1}^{2}\sum_{\alpha,\beta=1}^{4} u_{k\alpha,l\beta} |k\rangle_S\langle l| \otimes |\alpha\rangle_B\langle\beta|$, then $U_{SB}^{-1} = U_{SB}^{\dagger} = \sum_{k,l=1}^{2}\sum_{\alpha,\beta=1}^{4} u^{*}_{k\alpha,l\beta} |l\rangle_S\langle k| \otimes |\beta\rangle_B\langle\alpha|$, and so,

$$\begin{aligned} J_i &= \sum_{k,l=1}^{2}\sum_{\alpha,\beta=1}^{4} u^{*}_{k\alpha,l\beta} \langle i|\beta\rangle_B \langle\alpha|1\rangle_B |l\rangle_S\langle k| \\ &= \sum_{k,l=1}^{2} u^{*}_{k1,li} |l\rangle_S\langle k| \quad for\ i = 1, 2, 3, 4. \end{aligned} \tag{3.17}$$

Using the technique mentioned above, we find out the Kraus operators (J_1, J_2, J_3, J_4) corresponding to the channel which is given by the inverse of the unitary U_{SB}.

3.4.2 Fidelity of the quantum state

We can now consider the entire episode in the following fashion,

$$\rho_{AB} \xrightarrow{GADC} \rho'_{AB} = \textstyle\sum_{i=1}^{4}(\mathbb{I} \otimes K_i)\rho_{AB}(\mathbb{I} \otimes {K_i}^{\dagger}).$$

$$\rho_{AB} \xrightarrow{GADC} \rho'_{AB} = \textstyle\sum_{i=1}^{4}(\mathbb{I} \otimes K_i)\rho_{AB}(\mathbb{I} \otimes {K_i}^{\dagger}) \xrightarrow{\mathcal{M}_{\mathcal{N}}} \rho''_{AB}$$

Basically, in this case, we consider a two qubit ancilla along with the initial quantum state, and let the whole 4-qubit state pass through the unitary dilation corresponding to the GADC in the following way,

$$\rho_{AB} \xrightarrow{GADC} \rho'_{AB} = Tr_{B'B''}[(\mathbb{I}_A \otimes U_{SB})^\dagger (\rho_{AB} \otimes \phi_{B'B''})(\mathbb{I}_A \otimes U_{SB})] \tag{3.18}$$

After obtaining the evolved state by tracing out the ancillary qubits, the process of initialization of ancilla is employed and let the whole system pass through the inverse unitary map which corresponds to the reverse effect of the channel action. In this case our job is to check the closeness of the state ρ_{AB} with ρ''_{AB} and compare it with the closeness of states ρ_{AB} andρ'_{AB}. In this direction, we consider the fidelity as a quantifier for closeness and it is given as,

$$F(\rho_i, \rho_f) \equiv Tr[\sqrt{(\rho_i)^{\frac{1}{2}} \rho_f (\rho_i)^{\frac{1}{2}}}] \tag{3.19}$$

It can be easily seen that implementation of the inverse map on the evolved system's state and the initialized ancilla, gives a high fidelity for a couple of values of the damping channel parameter. But this does not show any improvement when the corresponding correlation such as quantum entanglement or quantum steering is calculated as the procedure is another local operation which can never be possibly enhance any type of quantum correlation.

3.4.3 To preserve quantum correlation using unitary dialation

Now, following the procedure of finding the Kraus operator representation associated to the inverse of the unitary dilation of a quantum channel, mentioned in Sec. 3.4.1, here we obtain the set of Kraus operators $\{J_1^{(1)}, J_2^{(1)}, J_3^{(1)}, J_4^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, J_3^{(2)}, J_4^{(2)}\}$ corresponding to the unitaries given in Eq. (3.14) and (3.15)

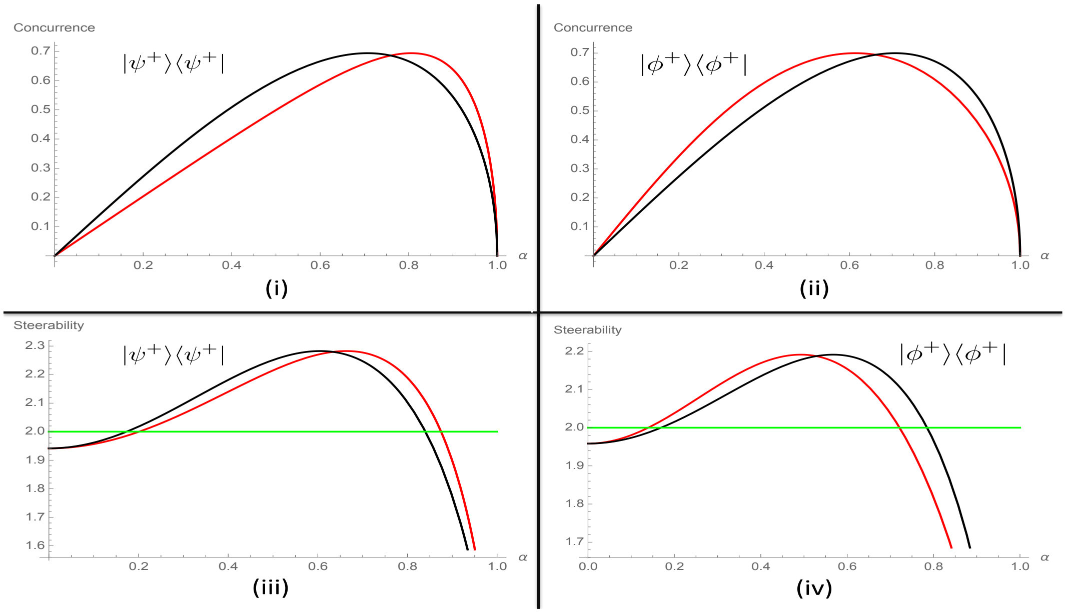

Figure 3.2: (i) Comparison of concurrence while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (ii) Comparison of concurrence while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$, (iii) Comparison of steerability while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (iv) Comparison of steerability while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$. In all the plots, Red curves denote the function with the POVM corresponding to inverse of unitary $U_{SB}^{(1)}$, being applied **before** the environmental interaction and Black curves denote the same without POVM, where Green lines in the plots (iii) and (iv) denote the limit of the violation of the ACHSH inequality. (For individual plots, ν and η have been kept fixed.)

respectively, which are illustrated below.

$$
\begin{aligned}
J_1^{(1)} &= \begin{pmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\eta\nu} \end{pmatrix} \\
J_2^{(1)} &= \begin{pmatrix} -\frac{\sqrt{\eta-\eta\nu}}{\sqrt{-\nu\eta+\eta+\nu}} & 0 \\ 0 & -\sqrt{\eta-\eta\nu} \end{pmatrix} \\
J_3^{(1)} &= \begin{pmatrix} 0 & -\sqrt{1-\eta} \\ 0 & 0 \end{pmatrix} \\
J_4^{(1)} &= \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{(\eta-1)(\nu-1)\nu}}{\sqrt{-\nu\eta+\eta+\nu}} & 0 \end{pmatrix},
\end{aligned} \tag{3.20}
$$

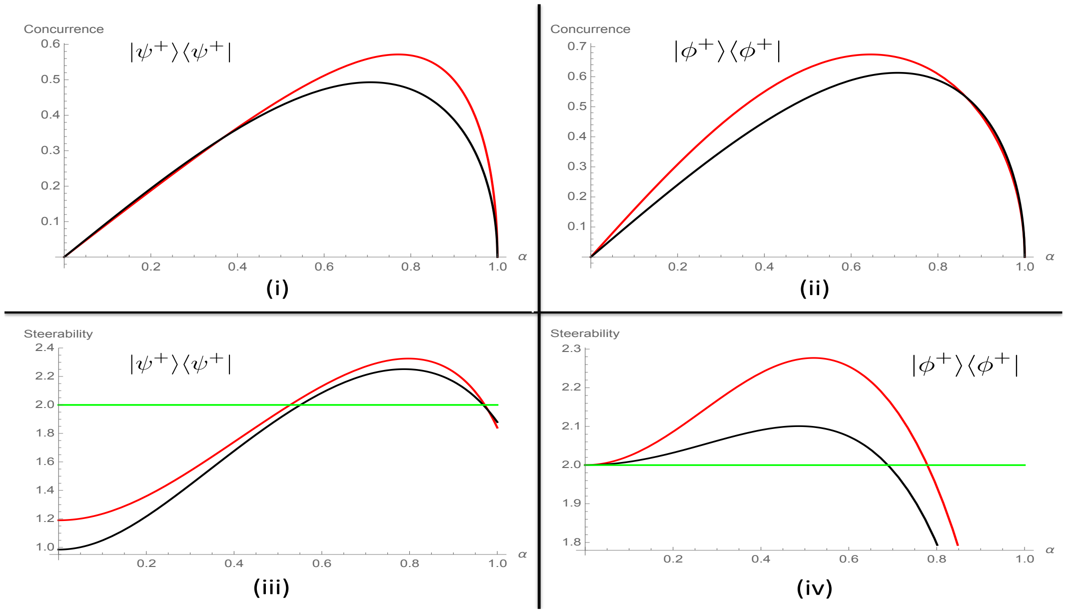

Figure 3.3: (i) Comparison of concurrence while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (ii) Comparison of concurrence while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$, (iii) Comparison of steerability while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (iv) Comparison of steerability while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$. In all the plots, Red curves denote the function with the POVM corresponding to inverse of unitary $U_{SB}^{(1)}$, being applied **after** the environmental interaction and Black curves denote the same without POVM, where Green lines in the plots (iii) and (iv) denote the limit of the violation of the ACHSH inequality. (For individual plots, ν and η have been kept fixed.)

and,

$$\begin{aligned}
J_1^{(2)} &= \begin{pmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\eta\nu} \end{pmatrix} \\
J_2^{(2)} &= \begin{pmatrix} -\frac{\sqrt{\nu}\sqrt{\eta-\eta\nu}}{\sqrt{\eta(\nu-1)+1}} & 0 \\ 0 & -\sqrt{\eta}\sqrt{1-\nu} \end{pmatrix} \\
J_3^{(2)} &= \begin{pmatrix} 0 & -\sqrt{1-\eta} \\ 0 & 0 \end{pmatrix} \\
J_4^{(2)} &= \begin{pmatrix} 0 & 0 \\ -\frac{\sqrt{(\eta-1)(\nu-1)}}{\sqrt{\eta(\nu-1)+1}} & 0 \end{pmatrix}.
\end{aligned} \tag{3.21}$$

Note that, $\sum_{i=1}^4 J_i^{(1)\dagger} J_i^{(1)} = \mathbb{I}$ and $\sum_{i=1}^4 J_i^{(2)\dagger} J_i^{(2)} = \mathbb{I}$. Let us now consider that

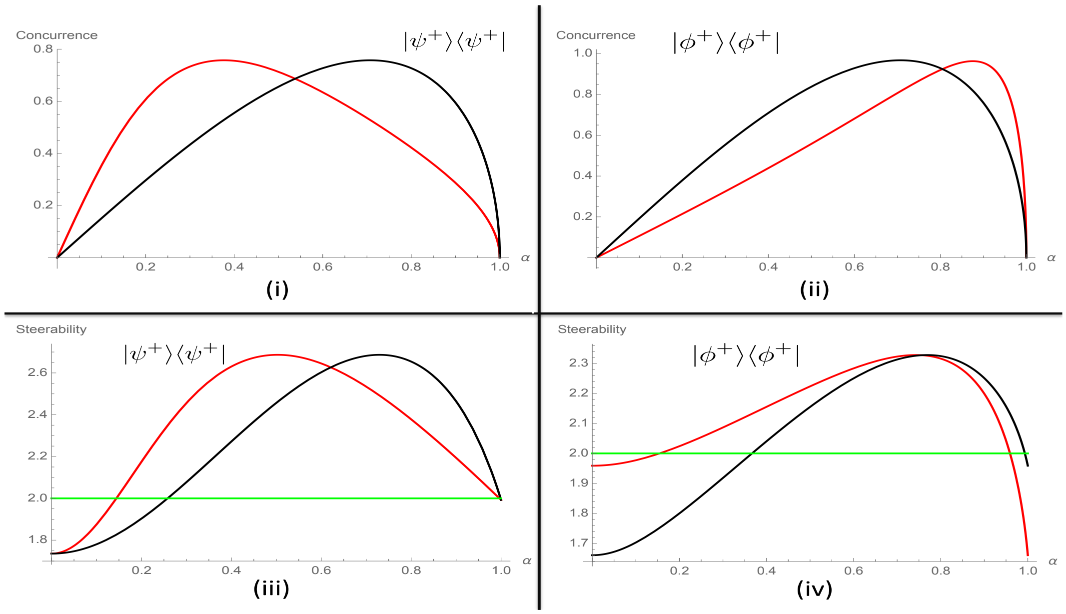

Figure 3.4: (i) Comparison of concurrence while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (ii) Comparison of concurrence while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$, (iii) Comparison of steerability while the initial state is taken to be the antiparallel state, $|\psi^+\rangle\langle\psi^+|$, (iv) Comparison of steerability while the initial state is taken to be the parallel state, $|\phi^+\rangle\langle\phi^+|$. In all the plots, Red curves denote the function with the POVM corresponding to inverse of unitary $U_{SB}^{(2)}$, being applied **before** the environmental interaction and Black curves denote the same without POVM, where Green lines in the plots (iii) and (iv) denote the limit of the violation of the ACHSH inequality. (For individual plots, ν and η have been kept fixed.)

one side (B, say) of the bipartite system AB is interacting with the environment through a GADC and hence we apply the selective POVM constructed from the individual element of the Kraus representation. In this scenario, we consider two different cases depending upon the order of application of the POVM.

Case I:

$$\rho_{AB} \xrightarrow{GADC} \rho'_{AB} = \sum_{i=1}^{4}(\mathbb{I}\otimes K_i)\rho_{AB}(\mathbb{I}\otimes {K_i}^{\dagger}).$$

$$\rho_{AB} \xrightarrow{POVM} \rho^{p(i)}_{AB} = (\mathbb{I}\otimes\{J_i^{\dagger}J_i\}^{\frac{1}{2}})\rho_{AB}(\mathbb{I}\otimes\{J_i^{\dagger}J_i\}^{\frac{1}{2}\dagger})$$
$$\xrightarrow{GADC} \rho^{pd(i)}_{AB} = \sum_{i=1}^{4}(\mathbb{I}\otimes K_i)\rho^{p}_{AB}(\mathbb{I}\otimes {K_i}^{\dagger}).$$

Case II:

$$\rho_{AB} \xrightarrow{GADC} \rho'_{AB} = \sum_{i=1}^{4}(\mathbb{I}\otimes K_i)\rho_{AB}(\mathbb{I}\otimes {K_i}^{\dagger}).$$

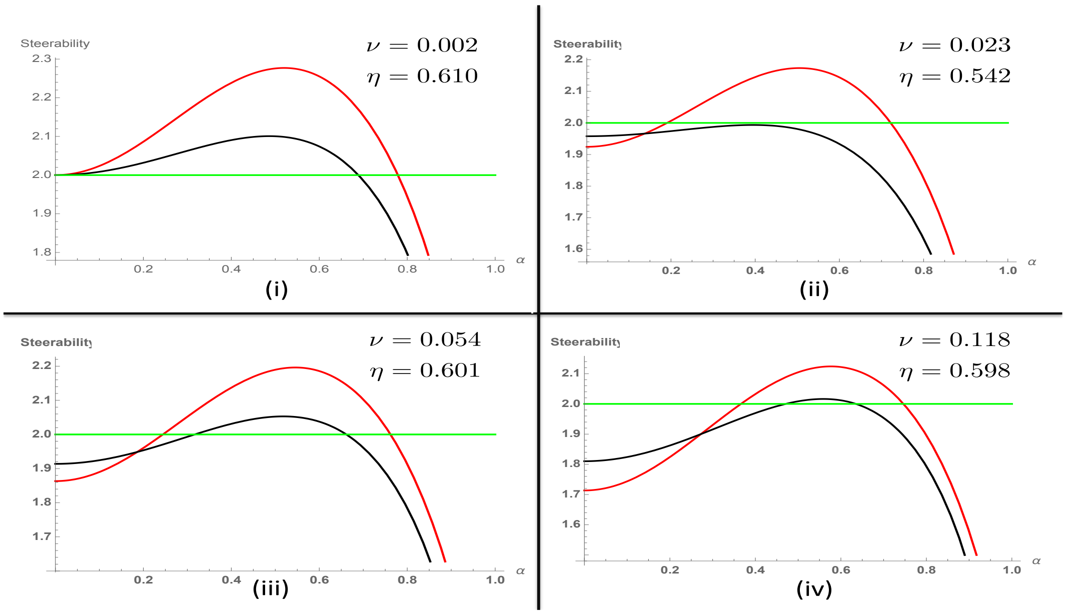

Figure 3.5: All the plots are showing the case (iv) of Fig. 3.3, i.e. the comparison of steerability in terms of the violation ACHSH inequality for different values of ν and η (the damping parameters of the given channel), when the initial state is considered to be a parallel state, $|\phi^+\rangle\langle\phi^+|$. The plot colors have their usual meaning.

$$\begin{aligned}\rho_{AB} &\xrightarrow{GADC} \rho'_{AB} = \textstyle\sum_{i=1}^{4}(\mathbb{I}\otimes K_i)\rho^{p}_{AB}(\mathbb{I}\otimes {K_i}^{\dagger})\\ &\xrightarrow{POVM} \rho^{dp(i)}_{AB} = (\mathbb{I}\otimes\{J_i^{\dagger}J_i\}^{\frac{1}{2}})\rho'_{AB}(\mathbb{I}\otimes{\{J_i^{\dagger}J_i\}^{\frac{1}{2}}}^{\dagger}).\end{aligned}$$

For Case I, we comparatively study the concurrence and steerability of the states ρ'_{AB} and $\rho^{pd(i)}_{AB}$, whereas in Case II, the similar protocol has been followed for the states ρ'_{AB} and $\rho^{dp(i)}_{AB}$. All the comparisons are demonstrated in the figures (3.2), (3.3) and (3.4).

From all the plots, improvement of quantum correlations on the application of POVM can be seen, over the sole interaction with the environment through GADC. This is a more general approach than the approach discussed in the Sec (3.3) and improvement can be seen over a larger range of values of the damping parameters η and ν in this case. Also, finding the suitable unitary matrix U_{SB} just by the method of basis expansion, one can identify the helpful POVM in protecting the quantum correlation, for a particular damping channel.

The motivation behind introducing the unitary dilation U_{SB} of a quantum channel Λ (acting on S) and the quantum channel Λ' (acting on S) whose unitary dilation being U^{-1}_{SB}, is to nullify the action of Λ. Such a scheme will act properly provided it can be guaranteed that $\Lambda(\rho_S) \otimes \sigma_B^{(1)}$ is closed to $U_{SB}[\Lambda(\rho_S) \otimes \sigma_B^{(0)}]U_{SB}^{-1}$ for two fixed states $\sigma_B^{(0)}$ and $\sigma_B^{(1)}$ of B. Needless to say that such a condition can not be satisfied, in general. And hence, our method can only recover the quantum correlation of the state partially. There are decoherence controlling models in literature for the noisy channels obtained by solving the optical master equation for thermal bath [51,53]. But for the particular case of GADC, which is obtained from squeezed thermal bath, our method of protecting quantum correlation is an effective procedure.

3.5 Summary

In this chapter, we have dealt with the problem of preserving quantum correlations that are useful in different information processing tasks, under the action of a noisy environment. Here, we choose GADC as the environmental noise and check its effect on entanglement and steerability of an initial pure bipartite state. First, we have employed the technique of weak measurement and reversal and found that a certain amount of improvement could be achieved. But, it can also be seen that this improvement is restricted for some particular values of the damping parameters of the corresponding channel. We have next introduced another method for the preservation of correlations using a unitary dilation of the operator sum representation of the channel. Interestingly, as the choice of the unitary is not unique, it provides us the freedom to choose the inverse evolution of the unitary, and hence the Kraus operators according to our convenience. Choosing different unitaries and consequently their inverses gives us the scope to extend our scheme over a larger range of the damping parameters, thus improving the quality of preservation. Note that in the present chapter we have considered two particular unitaries corresponding the Kraus representation of GADC for the illustration of our approach. However, it is possible to construct other unitaries taking the conditions of orthogonality and normalization into account. As a future direction, this method can be applied to other noisy channels and the choice of this unitary can be made suitably in order to generate an optimal scheme for protecting quantum correlations under the action of different noisy environments.

In quantum information theory as the protection mechanism is important from probable noisy environments, it is also important to identify the particular type of correlation in hand, to use it for proper purposes. In the next chapter, the procedure of self-testing is discussed in the light of 1-SDI scenario. It deals with the identification of the particular state (up to some local unitary) from a given probability distribution.

Chapter 4

One-sided Device-independent Self-testing of any Pure Two-qubit Entangled State

We consider the problem of one-sided device-independent self-testing of any pure entangled two-qubit state based on steering inequalities which certify the presence of quantum steering. In particular, we note that in the 2-2-2 steering scenario (involving two parties, two measurement settings per party, and two outcomes per measurement setting), the maximal violation of a fine-grained steering inequality can be used to witness certain extremal steerable correlations, which certify all pure two-qubit entangled states. We demonstrate that the violation of the analogous Clauser-Horne-Shimony-Holt inequality of steering or the non-vanishing value of a quantity constructed using a correlation function called mutual predictability, together with the maximal violation of the fine-grained steering inequality, can be used to self-test any pure entangled two-qubit state in a one-sided device-independent way.

4.1 Introduction and motivation

Based on two assumptions, viz. no signalling and the validity of quantum theory, the device-independent (DI) certification of quantum devices is a relevant research direction in quantum information as well as in quantum foundations [70]. The DI approach has several applications, for example, in random number certification [71], cryptography [72], testing the dimension of Hilbert Spaces [73]. In Ref. [74], a DI scheme called self-testing was proposed to certify a Bell state (maximally entangled two-qubit state) up to local isometries. Moreover, it has been shown that nonlocal correlations which are extremal [1] can be used for self-testing as these extremal correlations can only be achieved by performing particular measurements on a unique pure quantum state (up to some local isometry) [76]. Thus, by observing these extremal quantum correlations, it is possible to identify the entangled state without any assumption on the physical systems, measurements or even on the dimension of the relevant Hilbert Space of the given quantum system.

In Ref. [77], the first criterion for robust self-testing of a singlet state (maximally entangled bipartite qubit state) in the DI scenario was proposed using the maximal violation of Bell-CHSH (Bell-Clauser-Horne-Shimony-Holt) inequality [78]. In Ref. [79], a family of Bell inequalities called tilted Bell-CHSH inequality was studied. A family of extremal nonlocal correlations which can be simulated by a pure two-qubit entangled state can be identified by the maximal violation of the tilted Bell-CHSH inequality. In Ref. [80, 81], it has been shown that any pure two-qubit entangled state can be self-tested in a fully DI way by using these extremal correlations. In Refs. [74, 82], criteria for DI certification of quantum system were proposed without using Bell inequalities. In Ref. [76], it has been shown that any pure two-qudit entangled state can be self-tested in the Bell scenario where Alice and Bob perform three and four d-outcomes measurements on their respective sides.

Steering inequalities [83, 84] which are analogous to Bell inequalities are used to certify the presence of steering. The violation of a steering inequality certifies the presence of entanglement in a one-sided device-independent (1SDI) way. This has implication for quantum information processing in which quantum steering

[1] An extremal quantum correlation in a given Bell scenario cannot be decomposed as a convex mixture of the other quantum correlations in that given Bell scenario. Note that, there exists extremal quantum correlations which do not violate any Bell's inequality maximally in that Bell scenario [75].

has been used as a resource for 1SDI quantum key distribution and randomness generation [85, 86]. It should be noted that it is easier and more cost effective to implement these 1SDI tasks than to implement the completely DI tasks in laboratory. It is thus very relevant and important to study the self-testing problem in the 1SDI framework. Recently, 1SDI self-testing of maximally entangled two-qubit state has been proposed [87, 88]. In this context, the maximal violation of a linear steering inequality [83] was shown to self-test the maximally entangled two-qubit state in a 1SDI way. Moreover, self-testing via quantum steering was shown to provide certain advantages over DI self-testing.

In this work, we are interested in the problem of self-testing of any pure two-qubit entangled state in the 1SDI scenario. For this purpose, we consider two steering inequalities, *viz.* the fine grained inequality (FGI) [19], whose maximum violation certifies that the shared state is pure two-qubit entangled state, and the analogous CHSH inequality for steering [20]. We demonstrate that the violation of the analogous CHSH inequality of steering together with the maximal violation of the fine-grained steering inequality can be used to self-test any pure entangled two-qubit state in a 1SDI way. We further propose another scheme for 1SDI self-testing of any pure two-qubit entangled state in which the non-vanishing value of a quantity constructed using a correlation function called "mutual predictability" together with the maximal violation of the fine-grained steering inequality is used.

4.2 Self-testing via quantum steering

Device independent self-testing of quantum states through the violation of a Bell inequality occurs only for pure entangled states because it requires the observation of an extremal nonlocal correlation [75]. Therefore, in the self-testing problem, certifying a particular pure entangled state is of interest.

Suppose Alice and Bob want to self-test a particular pure entangled state $|\tilde{\psi}\rangle_{AB} \in \mathcal{H}'_A \otimes \mathcal{H}_B$ from the steerable assemblage arising from the state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ and measurement operators $\{M_{a|x}\}_{a,x}$ on Alice's side in the aforementioned 1SDI scenario. Then the assemblage self-tests the pure entangled state

$|\tilde{\psi}\rangle_{AB}$ if there exists an isometry on Alice' side Φ: $\mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \mathcal{H}'_A$ such that

$$\begin{aligned} \Phi(|\psi\rangle_{AB}) &= |junk\rangle_A \otimes |\tilde{\psi}\rangle_{AB}\,, \\ \Phi(M_{a|x} \otimes \mathbb{I}\,|\psi\rangle_{AB}) &= |junk\rangle_A \otimes (\tilde{M}_{a|x} \otimes \mathbb{I})\,|\tilde{\psi}\rangle_{AB}\,, \end{aligned} \tag{4.1}$$

where $|junk\rangle_A \in \mathcal{H}_A$ and $\{\tilde{M}_{a|x}\}_{a,x}$ are the measurement operators acting on the Hilbert space $\mathcal{H}'_A$. In Ref. [87], self-testing of maximally entangled two-qubit state based on the steerable assemblage arising from the two-setting steering scenario was proposed.

Analogous to the DI self-testing scheme based on the maximal violation of a Bell inequality, the measurement correlations $p(ab|xy) = Tr(\Pi_{b|y}\sigma_{a|x})$ arising from the assemblage can also be used to self-test the particular pure entangled state. In Refs. [87, 88], self-testing of the maximally entangled two-qubit state based on the maximal steerable correlation was proposed through the maximal violation of a steering inequality. That is, it was shown that the maximal violation of the linear steering inequality [83],

$$\langle A_0\sigma_z\rangle + \langle A_1\sigma_x\rangle \leq \sqrt{2}, \tag{4.2}$$

self-tests a maximally entangled two-qubit state. Here $\langle A_x B_y\rangle = \sum_{a,b}(-1)^{a\oplus b}p(ab|xy)$ with B_y being equal to σ_z or σ_x.

4.3 A 1SDI Self-testing Scheme for any pure bipartite qubit entangled state

Here we consider a steering scenario as described in Sec. 2, in which Alice performs two black-box dichotomic measurements and Bob performs two qubit measurements in mutually unbiased bases for self-testing any pure two-qubit entangled state in a 1SDI way. For this steering scenario, a necessary and sufficient condition for quantum steering in the form of steering inequality has been proposed in Eq. (2.7). Our self-testing scheme for certifying any pure bipartite qubit entangled state is based on the violation of the CFFW inequality with the maximal violation of another steering inequality, *i.e.*, the fine-grained inequality (FGI), given in Eq. (2.8).

Interestingly, we will now demonstrate that the maximum violation of the FGI by a shared two-qubit quantum state is achieved if and only if the shared state is any pure (maximally or non-maximally) entangled two-qubit state.

4.3.1 Lemma 1

Suppose the trusted party, Bob (B), performs projective qubit measurements in mutually unbiased bases (as we will consider CFFW or ACHSH inequality for steering later) corresponding to the operators $B_0 = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $B_1 = |+\rangle\langle +| - |-\rangle\langle -|$, where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis and $\{|+\rangle, |-\rangle\}$ is another orthonormal basis given by, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then the correlation violates FGI maximally if and only if the two-qubit state has the form,

$$|\psi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle \qquad 0 < \theta < \frac{\pi}{2} \tag{4.3}$$

up to local unitary transformations and Alice (A) performs projective measurements corresponding to the two operators given by, $A_0 = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $A_1 = \cos 2\theta(|0\rangle\langle 0| - |1\rangle\langle 1|) + \sin 2\theta(|+\rangle\langle +| - |-\rangle\langle -|)$ (or their local unitary equivalents).

Proof

It can be checked by simple calculation that, for B_0, B_1 mentioned above, if Alice and Bob share the state $|\psi(\theta)\rangle$ given by Eq.(4.3) and Alice performs projective measurements corresponding to the operators $A_0 = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $A_1 = \cos 2\theta(|0\rangle\langle 0| - |1\rangle\langle 1|) + \sin 2\theta(|+\rangle\langle +| - |-\rangle\langle -|)$ (or their local unitary equivalents), then the value of left hand side of FGI is 2. This numerical value 2 is the maximum violation of FGI since the algebraic maximum of left hand side of FGI is 2.

Violation of any steering inequality implies that the shared state is steerable and, hence, entangled. Therefore, the shared bipartite qubit state giving rise to the maximum violation of FGI is either a pure or a mixed entangled state. Note that the left hand side of FGI is the sum of two conditional probabilities and the magnitude of its maximum quantum violation is 2. Hence, maximum quantum violation of FGI implies that each of the two conditional probabilities appearing in the left hand side of FGI given by Eq. (2.8) is equal to 1, i.e. $P(b_{B_0} \mid a_{A_0}) = 1$ and $P(b_{B_1} \mid a_{A_1}) = 1$. This implies that, by performing measurements of the observables corresponding to the operators A_0 and A_1 on her particle, Alice can predict with certainty the outcomes of Bob's two different measurements of the two observables corresponding to the operators B_0 and B_1, respectively, without interacting with Bob's particle, where B_0 and B_1 are two mutually unbiased qubit measurements as described earlier. This is nothing but the EPR paradox[2] [6].

Suppose, $\rho_{0|0}$ and $\rho_{1|0}$ denote the normalised conditional states at Bob's end when Alice gets outcome $a = 0$ and $a = 1$, respectively, by performing the measurement A_0. The states $\rho_{0|0}$ and $\rho_{1|0}$ should be the eigenstates of the operator B_0 as maximum quantum violation of FGI implies Bob's conditional probability $P(b_{B_0} \mid a_{A_0}) = 1$ for $a = 0$ and for $a = 1$. Again, suppose, $\rho_{0|1}$ and $\rho_{1|1}$ denote the normalised conditional states at Bob's end while Alice gets the outcome $a = 0$ and $a = 1$, respectively, by performing the measurement A_1. Following similar arguments, it can be shown that the states $\rho_{0|1}$ and $\rho_{1|1}$ should be the eigenstates of the operator B_1. Hence, all the four conditional states at Bob's side $\rho_{0|0}$, $\rho_{1|0}$, $\rho_{0|1}$ and $\rho_{1|1}$ are pure. If the shared state between Alice and Bob is a pure entangled state, then it has been shown that the four conditional states at Bob's side are pure [89]. Now in the following we prove that these pure steerable assemblages can not be obtained from any mixed state, hence showing that the maximal violation of FGI can be obtained only from a pure entangled state.

Let us assume that $\sigma_{0|0}$, $\sigma_{1|0}$, $\sigma_{0|1}$ and $\sigma_{1|1}$ denote the elements of assemblage prepared at Bob's side which corresponds to maximum violation of FGI. Each element $\sigma_{a|x}$ of the assemblage is related to the normalised conditional state $\rho_{a|x}$ at Bob's side through the relation given by, $\sigma_{a|x} = p(a|x)\rho_{a|x}$, where $p(a|x)$ is

[2]EPR paradox occurs when Alices pair of local quantum measurements prepare two different set of quantum states at Bobs end which are eigenstates of two noncommuting observables.

the conditional probability of getting the outcome a when Alice performs the measurement A_x; $x \in \{0,1\}$; $a \in \{0,1\}$. Since each of the conditional states $\rho_{a|x}$ are pure, they cannot be expressed as convex mixture of two different states. Moreover, these conditional states are associated with the steerable assemblage $\{\sigma_{a|x}\}_{a,x}$ giving rise to maximum violation of FGI. Hence, steerable weight of the assemblage $\{\sigma_{a|x}\}_{a,x}$ giving rise to maximum violation of FGI must be 1.

According to Lewenstein-Sanpera decomposition, any bipartite qubit state ρ has a unique decomposition in the form [90]

$$\rho = \mu\rho^{ent}_{pure} + (1-\mu)\rho^{sep}, \tag{4.4}$$

where $0 \leq \mu \leq 1$, ρ^{ent}_{pure} is a bipartite qubit pure entangled state and ρ^{sep} is a bipartite qubit separable state. Here, $\mu = 1$ implies that ρ is a pure entangled state and $\mu = 0$ implies that ρ is separable. So, for any bipartite qubit mixed entangled state ρ^m, $\mu \neq 1$ and $\mu \neq 0$, i. e., $0 < \mu < 1$. Consider $\{\sigma^m_{a|x}\}_{a,x}$ denotes an arbitrary assemblage produced from ρ^m when Alice performs measurements $\{M_{a|x}\}_{a,x}$. Here $\sigma^m_{a|x} = Tr_A[(M_{a|x} \otimes \mathbb{I})\rho^m]$, $\forall \sigma^m_{a|x} \in \{\sigma^m_{a|x}\}_{a,x}$. Since ρ^m can always be expressed in the form given in Eq. (4.4),

$$\rho^m = \tilde{\mu}\tilde{\rho}^{ent}_{pure} + (1-\tilde{\mu})\tilde{\rho}^{sep}, \tag{4.5}$$

with $0 < \tilde{\mu} < 1$, we have for all a and x

$$\begin{aligned} \sigma^m_{a|x} &= Tr_A[(M_{a|x} \otimes \mathbb{I})(\tilde{\mu}\tilde{\rho}^{ent}_{pure} + (1-\tilde{\mu})\tilde{\rho}^{sep})] \\ &= \tilde{\mu} Tr_A[(M_{a|x} \otimes \mathbb{I})\tilde{\rho}^{ent}_{pure}] \\ &+ (1-\tilde{\mu}) Tr_A[(M_{a|x} \otimes \mathbb{I})\tilde{\rho}^{sep}] \\ &= \tilde{\mu}\tilde{\sigma}^{ent}_{pure_{a|x}} + (1-\tilde{\mu})\tilde{\sigma}^{sep}_{a|x}, \end{aligned} \tag{4.6}$$

$\tilde{\rho}^{ent}_{pure}$ is a bipartite qubit pure entangled state, $\tilde{\rho}^{sep}$ is a bipartite qubit separable state, $\tilde{\sigma}^{ent}_{pure_{a|x}}$ is an element of the assemblage $\{\tilde{\sigma}^{ent}_{pure_{a|x}}\}_{a,x}$ produced from the bipartite qubit pure entangled state $\tilde{\rho}^{ent}_{pure}$ when Alice performs measurements $\{M_{a|x}\}_{a,x}$ and $\tilde{\sigma}^{sep}_{a|x}$ is an element of the assemblage $\{\tilde{\sigma}^{sep}_{a|x}\}_{a,x}$ produced from the bipartite qubit separable state $\tilde{\rho}^{sep}$ when Alice performs measurements $\{M_{a|x}\}_{a,x}$. Since, steerable weight is a convex steering monotone [33] we have

$$\begin{aligned} & SW(\{\sigma^m_{a|x}\}_{a,x}) \\ & = SW(\tilde{\mu}\{\tilde{\sigma}^{ent}_{pure_{a|x}}\}_{a,x} + (1-\tilde{\mu})\{\tilde{\sigma}^{sep}_{a|x}\}_{a,x}) \\ & \leq \tilde{\mu} SW(\{\tilde{\sigma}^{ent}_{pure_{a|x}}\}_{a,x}) + (1-\tilde{\mu}) SW(\{\tilde{\sigma}^{sep}_{a|x}\}_{a,x}), \end{aligned} \tag{4.7}$$

where $SW(.)$ denotes the steerable weights of the corresponding assemblages. As, $\{\tilde{\sigma}^{sep}_{a|x}\}_{a,x}$ is the assemblage produced from a separable state, it is an unsteerable assemblage and hence $SW(\{\tilde{\sigma}^{sep}_{a|x}\}_{a,x}) = 0$ [33]. On the other hand, $0 \leq SW(\{\tilde{\sigma}^{ent}_{pure_{a|x}}\}_{a,x}) \leq 1$. Hence, from Eq.(4.7) we get

$$SW(\{\sigma^m_{a|x}\}_{a,x}) \leq \tilde{\mu} < 1 \tag{4.8}$$

We have, therefore, proved that steerable weight of an arbitrary bipartite qubit mixed entangled state cannot be equal to 1. On the other hand, it has been shown that the steerable weight of the assemblage produced by performing appropriate measurements on an arbitrary pure entangled state is equal to 1 [32]. Since maximum violation of FGI implies that the corresponding assemblage have steerable weight equal to 1, the maximum violation of FGI is achieved only if the shared bipartite qubit state between Alice and Bob is a pure entangled state.

The general form of any bipartite qubit pure entangled state is given by, $|\psi_p\rangle = |\psi(\theta)\rangle = \cos\theta |00\rangle + \sin\theta |11\rangle$, where $0 < \theta < \frac{\pi}{2}$, up to local unitary transformations. Maximum violation of FGI implies that $P(b_{B_0} \mid a_{A_0}) = 1$ and $P(b_{B_1} \mid a_{A_1}) = 1$. Let us assume that $a = 0$ and $b = 0$. In this case, it can be easily checked that, for B_0 mentioned above, $P(b_{B_0} \mid a_{A_0}) = 1$ for the aforementioned state $|\psi_p\rangle$ implies that Alice performs projective measurement of the observable corresponding to the operator $A_0 = |0\rangle\langle 0| - |1\rangle\langle 1|$. Moreover, it can also be checked that $P(b_{B_1} \mid a_{A_1}) = 1$ using aforementioned B_1 by the state $|\psi_p\rangle$ implies that Alice performs projective measurement of the observable corresponding to the operator $A_1 = \cos 2\theta(|0\rangle\langle 0| - |1\rangle\langle 1|) + \sin 2\theta(|+\rangle\langle +| - |-\rangle\langle -|)$. Hence the claim.

For other possible outcomes (a and b) the proof is similar.

4.3.2 Lemma 2

The maximal violation of FGI, i.e., 2 is obtained in our 1SDI scenario where Bob performs the two mutually unbiased qubit measurements corresponding to the operators given in the previous Lemma 1. Let this maximal violation be achieved from the assemblage arising from the pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ (where the dimension of $\mathcal{H}_B$ is 2) and measurement operators $\{M_{a|x}\}_{a,x}$ on Alice's side. Then there exists an isometry on Alice's side Φ: $\mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}'_A$, where the dimension of $\mathcal{H}'_A$ is 2, such that

$$\begin{aligned} \Phi(|\psi\rangle_{AB}) &= |junk\rangle_A \otimes |\psi(\theta)\rangle_{AB}\,, \\ \Phi(M_{a|x} \otimes \mathbb{I}\,|\psi\rangle_{AB}) &= |junk\rangle_A \otimes (\tilde{M}_{a|x} \otimes \mathbb{I})\,|\psi(\theta)\rangle_{AB}\,, \end{aligned} \tag{4.9}$$

where $|junk\rangle_A \in \mathcal{H}_A$, $|\psi(\theta)\rangle_{AB}$ is given by Eq. (4.3) and $\{\tilde{M}_{a|x}\}_{a,x}$ are the measurement operators on Alice's side corresponding to the observables given in Lemma 1.

Proof

Let us recall a lemma given in Ref. [91] which states that, given two Hermitian operators A_0 and A_1 with eigenvalues ± 1 acting on a Hilbert space $\mathcal{H}$, there is a decomposition of $\mathcal{H}$ as a direct sum of subspaces $\mathcal{H}_i$ of dimension $d \leq 2$ each, such that both A_0 and A_1 act within each $\mathcal{H}_i$, that is, they can be written as $A_0 = \oplus_i A_0^i$ and $A_1 = \oplus_i A_1^i$, where A_0^i and A_1^i act on $\mathcal{H}_i$.

In general, in our steering scenario, any shared bipartite state lies in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ where the dimension of $\mathcal{H}_A$ (the untrusted side) is 'd' and the dimension of $\mathcal{H}_B$ (the trusted side) is 2. From the above lemma it follows that the measurement observables acting on $\mathcal{H}_A$ act within each subspace $\mathcal{H}_A^i$ with dimension $d \leq 2$ of $\mathcal{H}_A$. Note that

$$\mathcal{H}_A \otimes \mathcal{H}_B = (\oplus_i \mathcal{H}_A^i) \otimes \mathcal{H}_B \simeq \oplus_i (\mathcal{H}_i \otimes \mathcal{H}_B). \tag{4.10}$$

It follows that the pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ and the measurement operators $\{M_{a|x}\}_{a,x}$ that give rise to the maximal violation of FGI can be decomposed as

$$|\psi\rangle_{AB} = \oplus_i \sqrt{q_i}\, |\psi\rangle^i_{AB}\,, \tag{4.11}$$

with $\sum_i q_i = 1$, where $|\psi\rangle^i_{AB}$ is a 2×2 pure state and

$$M_{a|x} = \oplus_i M^i_{a|x}, \tag{4.12}$$

where $M^i_{a|x}$ is an operator acting on $\mathcal{H}^i_A$ of $d = 2$.

From our Lemma 1, it follows that each $|\psi\rangle^i_{AB}$ in Eq. (4.11) should be of the following form:

$$|\psi\rangle^i_{AB} = \cos\theta\,|2i, 0\rangle + \sin\theta\,|2i+1, 1\rangle\,, \tag{4.13}$$

and for $x = 0$, $M^i_{a|x}$ in Eq. (4.12) are given by $M^i_{0|0} = |2i\rangle\langle 2i|$ and $M^i_{1|0} = |2i+1\rangle\langle 2i+1|$.

Alice can append a local ancilla qubit prepared in the state $|0\rangle'_A$ and look for a local isometry $\varPhi$ such that

$$(\varPhi \otimes \mathbb{I})\,|\psi\rangle_{AB}\,|0\rangle'_A = |junk\rangle_A \otimes |\psi(\theta)\rangle_{A'B}\,, \tag{4.14}$$

where $|junk\rangle_A$ is the junk state and $|\psi(\theta)\rangle_{A'B}$ is the state given by Eq. (4.3). This can be achieved for $\varPhi$ defined by the map

$$\varPhi\,|2k, 0\rangle_{AA'} \longmapsto |2k, 0\rangle_{AA'}\,, \tag{4.15}$$

$$\varPhi\,|2k+1, 0\rangle_{AA'} \longmapsto |2k, 1\rangle_{AA'}\,. \tag{4.16}$$

Thus, up to local isometry on Alice's side the maximal violation of FGI certifies any pure two-qubit entangled state in our 1SDI scenario, because any pure two-qubit entangled state can always be written in the form given by Eq. (4.3) following Schmidt decomposition [92, 93]. In order to identify which pure entangled two-qubit state has been certified, we consider violation of the CFFW inequality (2.7). For the state $|\psi(\theta)\rangle$ given by Eq.(4.3) with the aforementioned measure-

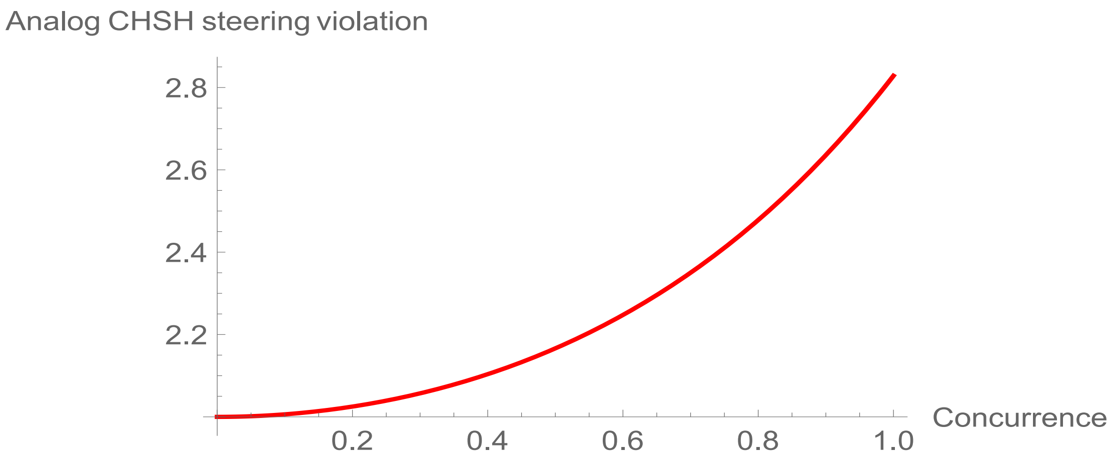

Figure 4.1: Along x-axis we plot concurrence of the state $|\psi(\theta)\rangle$ and along y-axis, we plot the LHS of the analogue CHSH steering inequality.

ment settings on Alice's and Bob's side, violation of the CFFW inequality is given by,

$$p = \sqrt{4\sin^4(\theta) + \sin^4(2\theta)} + \sqrt{\sin^4(2\theta) + 4\cos^4(\theta)}. \tag{4.17}$$

Note that concurrence which is a measure of entanglement [94] of the state $|\psi(\theta)\rangle$ given by Eq. (4.3) turns out to be $C = \sin 2\theta$. It is now readily seen that the violation of CFFW inequality (2.7) for the state $|\psi(\theta)\rangle$ given by Eq. (4.3) and concurrence of this family of states are functions of θ. Hence, one can easily find out the concurrence of the pure two-qubit state, which is self-tested by the maximum violation of FGI, from the violation of the CFFW inequality. In other words, from the violation of the CFFW inequality one can particularly identify which pure two-qubit entangled state has been self-tested.

The variation of the violation 'p' with concurrence 'C' is monotonic and continuous and it is shown in Fig. (4.1). Hence, from the violation of the CFFW inequality given by Eq. (2.7), one can certify whether the pure two-qubit entangled state is maximally entangled or non-maximally entangled. Moreover, from the violation of CFFW inequality using the plot presented in Fig. (4.1), one can find out which particular pure two-qubit entangled state has been self-tested. Therefore, we can state the following self-testing result.

Result. The maximal violation of FGI self-tests any pure two-qubit entangled state. On the other hand, magnitude of the violation of CFFW inequality for the measurements that give rise to the maximal violation of FGI certifies the amount of entanglement of the self-tested pure two-qubit entangled state, i.e., magnitude of the violation of CFFW inequality for the measurements that give rise to the maximal violation of FGI identifies which particular pure two-qubit entangled state has been self-tested by FGI.

Note that, for the maximally entangled state, our scheme for 1SDI self-testing reduces to the 1SDI self-testing scheme based on the maximal violation of the steering inequality given in Eq. (4.2).

For the above self-testing scheme, the full knowledge of the measurement correlations $p(ab|xy)$ is needed. We will now propose a scheme which does not require the full knowledge of the measurement correlations. This scheme is based on the maximal violation of FGI together with non-vanishing value of a correlation function called "mutual predictability" which has been used for constructing entanglement witness [95] and steering inequality [96]. For the dichotomic observables A_x and B_y on Alice's and Bob's side, respectively, the mutual predictability is given by

$$C_{A_xB_y} = p(a=0, b=0|x,y) + p(a=1, b=1|x,y) \tag{4.18}$$

Let us now consider the following quantity in the context of our 1SDI scenario. This quantity is defined in terms of mutual predictability as follows:

$$E = \min\{C_{A_0B_0}, C_{A_1B_1}\}. \tag{4.19}$$

Note that for the pure state given by Eq. (4.3) and the measurements that are specified in Lemma 1, the above quantity is related to concurrence of the state $|\psi(\theta)\rangle$, given by $C = \sin 2\theta$, as $E = \dfrac{1+C^2}{2}$. Hence, the quantity E (or, $2E-1$) is a monotonic function of concurrence of the state $|\psi(\theta)\rangle$. Therefore, we can state another self-testing scheme below.

Result. The maximal violation of FGI self-tests any pure two-qubit entangled state. On the other hand, nonvanishing value of the quantity $2E-1$, where E defined in Eq. (4.19), for the measurements that give rise to the maximal violation of FGI certifies concurrence of the self-tested pure two-qubit entangled state, i. e., the non-vanishing value of the quantity $2E-1$ for the measurements that give rise to the maximal violation of FGI identifies which particular pure two-qubit entangled state has been self-tested by FGI.

It is also interesting to note that for the above scheme, it is not necessary to assume that the trusted party performs measurements in mutually unbiased bases. In fact we can assume projective measurements of arbitrary pair of noncommuting qubit observables at trusted party's side. In this case also maximum violation of FGI implies EPR paradox [6], which is only possible if the shared state is any pure two-qubit entangled state [89]. So this scheme can also be used for self-testing in the dimension-bounded steering scenario [97] where only the Hilbert-space dimension of measurements of the trusted party is assumed.

4.4 Summary

Quantum steering which is a weaker form of quantum inseparabilities compared to Bell-nonlocality, certifies the presence of entanglement in a 1-sided device-independent scenario. This method for certification of entanglement has advantages over entanglement certification methods based on Bell nonlocality and standard entanglement witnesses. Motivated by this, recently, 1-sided device-independent self-testing of the maximally entangled two-qubit state was proposed by Supic et. al. [87] and Gheorghiu et. al. [88] via quantum steering.

In this work, we have proposed two schemes to self-test any pure (maximally or non-maximally) bipartite qubit entangled state up to some local unitary, in a 1SDI way via quantum steering. One of our schemes is based on two different steering inequalities,

i) Fine-grained steering inequality (FGI) [19] and

ii) analogous CHSH inequality for steering [20].

We have shown that the violation of the analogous CHSH inequality for steering together with the maximal violation of FGI self-tests any pure two-qubit entangled state. In another scheme, we have demonstrated that the non-vanishing value of a quantity constructed from a correlation function called mutual predictability together with the maximal violation of FGI can be used to self-test any pure two-qubit entangled state in the dimension-bounded steering scenario.

This chapter deals with the identification of a pure entangled state based on the quantum correlation in hand, i.e. when a probability distribution is provided. One of the most fundamental problems of quantum information theory is the separability problem. The motivation of this kind of a problem is to identify if an arbitrary quantum state is entangled or not. This becomes a computationally hard problem when the system dimension is greater than six. In the next chapter, a protocol has been proposed, which detects bipartite qubit entangled states with minimal resources.

Chapter 5

Universal detection of entanglement in two-qubit states using only two copies

We revisit the problem of detection of entanglement of an unknown two-qubit state using minimal resources. Using weak values and just two copies of an arbitrary two-qubit state, we present a protocol where a post-selection measurement in the computational basis provides enough information to identify if the state is entangled or not. Our protocol enables complete state identification with a single-setting post-selection measurement on two copies of the state. It follows that by restricting to pure states, the global interaction required for determining the weak values can be realized by local operations. We further show that our protocol is robust against errors arising from inappropriate global interactions applied during weak value determination.

5.1 Introduction and motivation

Ever since the coinage of the word "entanglement" by Schrödinger in 1935 closely following the work of Einstein, Podolsky and Rosen (EPR) [6], discussion and debate about its nature and manifestation has continued to remain one of the most engaging issues in modern physics. In present times, entanglement is regarded as the primary building block of quantum correlations, leading to landmark discoveries in quantum information science [9]. Numerous protocols have already been suggested, which use these correlations as resource and result in improvements, which no classical resource could achieve [1–5,98]. It has been realized [10,11] that the nonlocal quantum correlations responsible for steering and Bell-violation cannot exist without the presence of entanglement. As a result of this, identification and quantification of quantum correlations, have become a topic of cutting edge research in various inter-disciplinary areas of physics, mathematics and computer science, as well.

In quantum information theory, the way of identifying entanglement in a given bipartite state is through the *separability criterion* [99,100]. Though this criterion is also helpful in quantification of entanglement [101, 102], it is measurable only when full knowledge of the state is available. Such knowledge requires state tomography [103], which is expensive in terms of resources required. On the other hand, there are methods based on direct measurement of observables (which are single setting measurements) such as *entanglement witnessing* [104–106] which have been experimentally realized [107, 108]. In addition, other schemes have been recently proposed, such as *self-testing* protocols as discussed in Chapter (4), which can identify individual entangled states giving rise to particular correlations in a given scenario [77, 80, 87, 109]. However, all such methods suffer from the drawback of non-universality. For instance, for every entanglement witness (EW) there exists a class of entangled states, which it cannot detect [105, 110]. This prevents the use of any single EW to detect all entangled states. It is pertinent to note here that arranging a higher number of measurement settings is an expensive resource in experiments.

Entanglement detection in two-qubit states has drawn renewed attention, as can be seen from several recent works [111–114]. The main motivation for the present study is to reduce the resources required for identifying entanglement, and here we concern ourselves with the task of identification of entanglement in an unknown quantum state. In this context, Yu et al. [115] constructed an observable acting on four copies of any two-qubit state, that could detect entanglement for certain classes of two-qubit states. Augusiak et al. [116] proposed the construction of an observable which acts on four copies of a two-qubit state and results in detection of all entangled states. Therefore, universal detection of entanglement could be done through measurement in a single setting, but the cost is to supply multiple copies of the state. Further work in this direction [117–120], has been performed to reduce the resources required for universal identification of entangled states. Girolami et al. [117] proposed a method for identifying quantum correlations in two-qubit states through measurement of seven observables on four copies, where the observables are local in the Alice-Bob cut (the two parties sharing the bipartite state). It has been shown [119, 120] that any universal entanglement detection scheme on a single copy of a state, has to be necessarily a state tomography process. Recently, the protocol in [116] was extended to the completely device independent scenario [121].

In the present work, we propose a protocol where universal detection of entanglement is possible in a single measurement setting on just *two* copies of any two-qubit state, using weak values. The idea of weak measurement was first proposed by Aharonov et al. in [18], to show that an experimental outcome outside the eigenvalue spectrum of an observable could be obtained if a sufficiently weak coupling of the system and the apparatus along with post-selection is employed. Weak measurements have been utilized in several interesting applications such as observations of spin Hall effect [59], trajectories of photons [122], direct measurement of the quantum wave function [63], and measurement of ultrasmall time delays of light [123]. The technique of weak measurement and reversal has also been used in the preservation of entanglement [124–127], teleportation fidelity [49] and steerability [50] through noisy channels. Detection of weak value has been found to be useful in observing geometric phase [128], non-Hermitian operators [129] and quantum state [130–132].

Here we show that our protocol of entanglement detection using weak values on two copies of an arbitrary two-qubit state results in complete identification of the state, i.e., state tomography, in the similar fashion as in [119, 120]. Note that, a number of attempts [118, 133–135] were made to measure concurrence [25, 136] of two qubit states through measurement of a single observable on two copies of the state. Although, for pure states [133] such observables could be found, only estimates could be given for mixed states [118, 134, 135]. In this regard, our result provides a solution to this problem, as complete identification of two-qubit states, obtained through our protocol, also imply measurement of concurrence for any two-qubit state using two copies. We further show that on restricting the set of states to just pure states, the weak interaction necessary in our protocol, can be realized through local operations on each of the qubits. Finally, we also show that our protocol is robust to errors arising from inappropriate choice of weak interaction between two copies of the two-qubit states.

5.2 Background

Any quantum state $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, can always be decomposed as,

$$\rho = \sum_{ijkl} p_{kl}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right| \tag{5.1}$$

where $\{\left|i\right\rangle\}_i$ denotes an orthonormal basis in each of the subsystem Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Using this decomposition, we can define the partial transpose of ρ, with respect to the subsystem B, in the following way,

$$\rho^{T_B} = \sum_{ijkl} p_{lk}^{ij} \left|i\right\rangle \left\langle j\right| \otimes \left|k\right\rangle \left\langle l\right| \tag{5.2}$$

Note that, we can similarly define ρ^{T_A} and $\rho^{T_B} = (\rho^{T_A})^T$, where $\bullet^T$ denotes transposition. Now, we can present the separability criteria, as mentioned in the previous section. Any two qubit state $\rho \in \mathcal{P}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$ is separable [116, 137, 138] if and only if,

$$det\ \left(\rho^{T_B}\right) \geq 0 \tag{5.3}$$

where $det(A)$ represents determinant of a matrix A. This criterion can also be linked to the quantification of entanglement in terms of concurrence [**?**].

Now the idea of weak measurement and weak values will be briefly illustrated here. In the theory of weak measurements [18, 21], the pointer system of the measuring apparatus is kept in an initial state $\phi_{in} \in \mathcal{P}_+(\mathcal{H}_E)$ and a quantum system is *pre-selected* in a state $\rho \in \mathcal{P}_+(\mathcal{H})$. Then, the joint system-pointer state is evolved through a *weak interaction* generated by a Hamiltonian $\epsilon H \otimes P_x$, where H is the Hamiltonian associated with the system, P_x is the momentum operator of the pointer system, and ϵ is a small positive number representing the *weakness* of interaction. Following this, a strong *post-selective* measurement is performed on the weakly evolved state of the system in a basis $\{|u_k\rangle\}_k$, where $|u_k\rangle \in \mathcal{H}$, which results in the pointer state $\phi_f^k \in \mathcal{P}_+(\mathcal{H}_E)$ for each k, where

$$\phi_f^k \approx \langle u_i| \, \rho \, |u_i\rangle \, e^{-i\epsilon \langle H\rangle_\rho^{(k)} \; P_x} \phi_{in} \; e^{i\epsilon \langle H\rangle_\rho^{(k)} \; P_x} \tag{5.4}$$

where $\langle H\rangle_\rho^{(k)}$ are the weak values, given by,

$$\langle H\rangle_\rho^{(k)} = \frac{tr\big[H\rho \, |u_k\rangle \, \langle u_k| \, \big]}{tr\big[\rho \, |u_k\rangle \, \langle u_k| \, \big]}. \tag{5.5}$$

Note that Eq. (5.4) can be derived only under the approximation that ϵ is very small. For measuring $\langle H\rangle_\rho^{(k)}$ certain properties of the position and momentum wave function of ϕ_f^k needs to be observed. As mentioned in Ref [139], these properties include shift in expectation value of the position and momentum wave function compared to their initial values, variance of the momentum wave function, rate of change of the position wave function, and strength of the weak interaction i e., ϵ. A detailed analysis on this technique is provided in section II of [139]. Also recently, real and imaginary parts of a weak value was detected by using Laguerre-Gaussian modes [140] in the pointer state. For a detailed discussion on weak values, refer to [21].

5.3 Entanglement witness via weak values using two copies of the state

In this section, we present a technique using weak values to detect entanglement of any two-qubit state, through a single projective measurement, i.e., measurement in a single setting. It was recently shown [141], that by suitable choice of Hamiltonian and post-selective measurement, weak values can be used to determine the concurrence of any *pure* two-qubit state. In this chapter, we generalize this idea to *any* two-qubit state. For this purpose, we consider only two copies of the two-qubit state in consideration.

Now, let us start by considering Alice and Bob share two copies of a two-qubit state $\rho \in \mathcal{P}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$. The most general form of the density matrix of a two-qubit state (mixed or pure) can be expressed in the following form,

$$\rho = \begin{pmatrix} p & u & v & w \\ u^* & q & x & y \\ v^* & x^* & r & z \\ w^* & y^* & z^* & s \end{pmatrix} \tag{5.6}$$

where, p, q, r and s are real, non-negative numbers summing up to 1, and u, v, w, x, y and z are complex numbers in general; u^* is the complex conjugate of u, etc. It should be noted that ρ is Hermitian. In addition to these conditions there is another constraint of positivity of the above matrix, which has to be satisfied by ρ to be a valid density matrix, but for our purpose here, we stick to the form given in Eq. (5.6).

5.3.1 The general case

We first consider the general case, where p, q, r and s are nonzero. As a result, the determinant of the partially transposed matrix of ρ can be written as,

$$\begin{aligned} det(\rho^{T_B}) \;=\; & pqrs\Big(\frac{uu^*zz^*}{pqrs} - \frac{uvy^*z^*}{pqrs} - \frac{uw^*xz}{pqrs} - \frac{u^*v^*yz}{pqrs} \\ & -\frac{u^*wx^*z^*}{pqrs} + \frac{vv^*yy^*}{pqrs} - \frac{vw^*x^*y}{pqrs} - \frac{v^*wxy^*}{pqrs} \\ & +\frac{ww^*xx^*}{pqrs} + \frac{uvw^*}{pqr} + \frac{u^*v^*w}{pqr} + \frac{uxy^*}{pqs} \\ & +\frac{u^*x^*y}{pqs} - \frac{uu^*}{pq} + \frac{vx^*z^*}{prs} + \frac{v^*xz}{prs} \\ & -\frac{vv^*}{pr} - \frac{xx^*}{ps} + \frac{wy^*z^*}{qrs} + \frac{w^*yz}{qrs} \\ & -\frac{ww^*}{qr} - \frac{yy^*}{qs} - \frac{zz^*}{rs} + 1\Big). \end{aligned} \tag{5.7}$$

It can be seen that the determinant in Eq. (5.7) is a polynomial of degree 4. In [142], it was shown that an n-th degree homogeneous polynomial function of the density matrix elements can be computed as the expectation value of a pair observables, which acts on n copies of the density matrix. This result was later on used by Augusiak et al. [116] to construct a single observable, acting on four copies of a two-qubit state, to compute the determinant in Eq. (5.7) for witnessing entanglement.

Our aim is to reduce the number of copies of the state required, and hence reduce the resources required for the process of witnessing. For this purpose we consider the technique of using weak measurement as in [141]. Note that, in Eq. (5.3), for detecting entanglement of the unknown state ρ, it is sufficient to know the sign of the determinant in Eq. (5.7). In other words it is enough to find the value of $(1/pqrs)\ det\rho^{T_B}$. We also found that, finding values of the following terms (and thereby, their complex conjugates) is sufficient to determine the value of $(1/pqrs)\ det\rho^{T_B}$:

$$\frac{u^*}{p}, \frac{u}{q}, \frac{z^*}{r}, \frac{z}{s}, \frac{v^*}{p}, \frac{y^*}{q}, \frac{v}{r}, \frac{y}{s}, \frac{w^*}{p}, \frac{x^*}{q}, \frac{x}{r}, \frac{w}{s}. \tag{5.8}$$

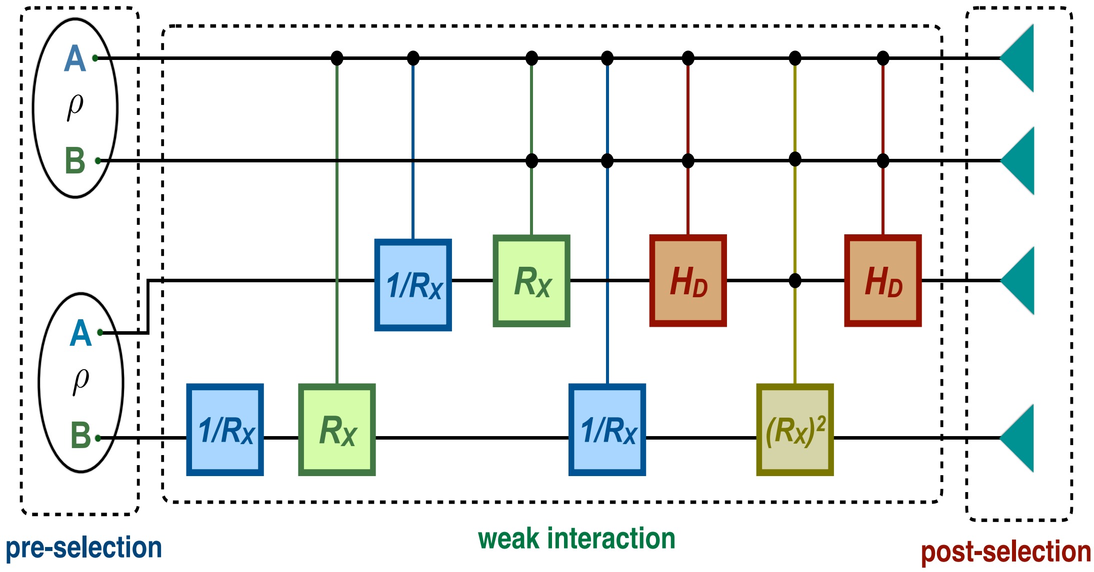

Figure 5.1: Circuit realization of entanglement detection through weak interaction. Here $R_X = e^{i\epsilon\sigma_x}$ represents rotation of the bloch vector about x-axis through an angle -2ϵ, and $H_D = |0\rangle\langle +| + |1\rangle\langle -|$ represents the Hadamard operation, where $|\pm\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle \pm |1\rangle\big)$.

Out of these 12 terms, it can be easily seen that 9 of them are independent. For example $\frac{u}{q}$, $\frac{z}{s}$ and $\frac{w^*}{p}$ can be expressed in terms of the remaining 9 terms. Note that, this latter condition does not result in any reduction of copies,i.e. resources required for the protocol.

We find that each of the terms in Eq. (5.8) can be seen as a weak value, as in Eq. (5.5), if we consider two copies of the state i.e. $\rho \otimes \rho \in \mathcal{P}_+((\mathcal{H}_A \otimes \mathcal{H}_B) \otimes (\mathcal{H}_A \otimes \mathcal{H}_B))$ and choose the Hamiltonian $H \in \mathcal{L}((\mathcal{H}_A \otimes \mathcal{H}_B) \otimes (\mathcal{H}_A \otimes \mathcal{H}_B))$ in an appropriate form, along with the post-selective measurement in the computational basis i.e., $\{|u_k\rangle\}_{k=1}^{16} = \{|0000\rangle, |0001\rangle, \ldots, |1111\rangle\}$. It turns out that a suitable form of H is the following,

$$\begin{aligned} H &= |00\rangle\langle 00| \otimes H_1 + |01\rangle\langle 01| \otimes H_1 \\ &\quad + |10\rangle\langle 10| \otimes H_2 + |11\rangle\langle 11| \otimes H_3 \end{aligned} \tag{5.9}$$

where,

$$
\begin{aligned}
H_1 &= \mathbb{I} \otimes \sigma_x &(5.10)\\
H_2 &= \sigma_x \otimes \mathbb{I} &(5.11)\\
H_3 &= \sigma_x \otimes \sigma_x &(5.12)
\end{aligned}
$$

with σ_x being the usual Pauli matrix along x-direction. Using the computational basis $\{|u_k\rangle\}_{k=1}^{16}$ and Eqs. (5.6) and (5.9), in Eq. (5.5), we find a list of weak values and the terms of Eq. (5.8), they correspond to,

$$
\begin{aligned}
\frac{u^*}{p} = \langle H\rangle^{(1)}_{\rho\otimes\rho} \quad &; \quad \frac{u}{q} = \langle H\rangle^{(2)}_{\rho\otimes\rho} \;\; ; \;\; \frac{z^*}{r} = \langle H\rangle^{(3)}_{\rho\otimes\rho} \; ;\\
\frac{z}{s} = \langle H\rangle^{(4)}_{\rho\otimes\rho} \quad &; \quad \frac{v^*}{p} = \langle H\rangle^{(9)}_{\rho\otimes\rho} \;\; ; \;\; \frac{y^*}{q} = \langle H\rangle^{(10)}_{\rho\otimes\rho} \; ;\\
\frac{v}{r} = \langle H\rangle^{(11)}_{\rho\otimes\rho} \quad &; \quad \frac{y}{s} = \langle H\rangle^{(12)}_{\rho\otimes\rho} \;\; ; \;\; \frac{w^*}{p} = \langle H\rangle^{(13)}_{\rho\otimes\rho} \; ;\\
\frac{x^*}{q} = \langle H\rangle^{(14)}_{\rho\otimes\rho} \quad &; \quad \frac{x}{r} = \langle H\rangle^{(15)}_{\rho\otimes\rho} \;\; ; \;\; \frac{w}{s} = \langle H\rangle^{(16)}_{\rho\otimes\rho} \qquad (5.13)
\end{aligned}
$$

Note that four weak values, generated out of the post selection measurement, are redundant. As a result they do not occur in the above equation. Therefore, it can be easily seen from Eq. (5.13), that our protocol leads to the determination of the sign of the determinant in Eq. (5.7), and as a results it would lead to universal entanglement detection for two-qubit quantum states. Note that in this protocol, detection of entanglement is made only through a single projective measurement setting, i.e., the post-selective measurement. Using the form of Hamiltonian given in Eq. (5.9), we find the unitary operator U, giving rise to weak interaction is given by,

$$
\begin{aligned}
U &= |0\rangle\langle 0| \otimes \mathbb{I} \otimes \mathbb{I} \otimes e^{-i\epsilon\sigma_x} + |10\rangle\langle 10| \otimes e^{-i\epsilon\sigma_x} \otimes \mathbb{I}\\
&\quad + |11\rangle\langle 11| \otimes e^{-i\epsilon\sigma_x\otimes\sigma_x} \qquad (5.14)
\end{aligned}
$$

In the above form, we can write $e^{-i\epsilon\sigma_x\otimes\sigma_x} = |+\rangle\langle +| \otimes e^{-i\epsilon\sigma_x} + |-\rangle\langle -| \otimes e^{i\epsilon\sigma_x}$. Note that, this represents a conditional unitary operation, conditioned on $\{|+\rangle, |-\rangle\}$

states. As a result we use Hadamard gate H_D, which flips states $\{|+\rangle, |-\}\rangle \leftrightarrow \{|0\rangle, |1\rangle\}$, to achieve the circuit realization of our protocol, as given in Fig. 5.1. Also note that, the decomposition of H in Eq.(5.9) is not unique, as it can also be chosen in any form where H_1, H_2 and H_3 resides in any of the diagonal blocks of the 16×16 matrix H. Moreover, as mentioned earlier, there are 9 independent terms in Eq. (5.8), which leads to 9 independent linear equations. Along with these equations, the constraint $p+q+r+s=1$, gives the exact solution for all the unknown quantities in the density matrix ρ, and hence our protocol results in complete identification of the two-qubit state i.e. state tomography. Henceforth, any standard method for finding the amount of entanglement like negativity [101,143] or concurrence [136] can be employed, and one can calculate how much entangled the given state is. In particular, the following quantity, which can be easily obtained from our protocol, can also used to estimate the amount of entanglement present in the state :

$$E(\rho) = max\{0, -det(\rho^{T_B})\}. \tag{5.15}$$

Thus we see our protocol not only serves as a technique to detect arbitrary two-qubit entangled state, but also as a protocol to measure entanglement.

5.3.2 Special cases

Let us now consider the special cases where at least one of the diagonal elements of ρ is zero. This scenario physically means receiving no signal on the pointer, for the corresponding measurement outcome, before the weak interaction is switched on. For example, if p is 0, no signal is received for outcome $|0000\rangle$, and similarly for q, r or s we check if no signal is received for the outcomes $|0101\rangle, |1010\rangle$ or $|1111\rangle$, respectively. We will show here that even for this case the same protocol, as described in Fig. 5.1, works. We now consider each case individually,

Case I

When $p = 0$, positivity of ρ demands u, v and w must also be 0. Similarly, when $s = 0$ we must have $w = y = z = 0$. As a result, in both of these cases, we have $det(\rho^{T_B}) = -xx^*qr$. In both of these cases, we first check if q or r is zero or not. If either of them is zero, we conclude ρ is separable. If not, we check if the weak value $\langle H\rangle^{(14)}_{\rho\otimes\rho}$ i e. x^*/q, is zero. If it is, then ρ is separable, otherwise it is entangled.

Case II

Similarly, when $q = 0$ or $r = 0$ we have $u = x = y = 0$ or $v = x = z = 0$, respectively. In both the cases we get $det(\rho^{T_B}) = -ww^*ps$. Following this, in a similar way as above, we check the values of p or s and subsequently, the weak value $\langle H\rangle^{(16)}_{\rho\otimes\rho}$ i e. w/s, to determine if ρ is entangled or not.

5.3.3 Implementing the protocol through Local Operations

In this section, we show that if we restrict the two-qubit state to be *pure* states only, we can realize the weak interaction through local operations on each of the qubits. Consider $\rho = |\Psi\rangle\langle\Psi|$, where $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. It can be easily seen that ρ is separable if and only if $ad - bc = 0$. In the notations of Eq. (5.6), we find $p = |a|^2, q = |b|^2, r = |c|^2, s = |d|^2, u = ab^*$ and $z = cd^*$. Therefore, for this case we modify our protocol, and choose the weak Hamiltonian in Eq (5.5) to be of the form,

$$H' = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes \sigma_x \tag{5.16}$$

It can be easily checked that the unitary operator corresponding to this Hamiltonian acts locally on all of the four qubits. Now, in the same way as in the previous section, we first check if either of p, q, r or s is zero i.e by checking if signal is received for outcomes $|0000\rangle, |0101\rangle, |1010\rangle$ or $|1111\rangle$. If one, or more of p, q, r and s are zero, it would imply the corresponding terms among a, b, c and d are also zero. Using these values we can easily check if $ad - bc = 0$. If none of p, q, r or s are zero, we check if the weak values $\langle H'\rangle^{(2)}_{\rho\otimes\rho} = \langle H'\rangle^{(4)}_{\rho\otimes\rho}$, i e. if $u/q = z/s$. If the equality holds then ρ is separable, otherwise it is entangled.

5.4 Robustness of the protocol

In real life experiments, errors are bound to occur. Here we show that, our protocol is robust against errors arising from inappropriate choice of weak interaction. Consider a situation, where an erroneous Hamiltonian of the form of H_e is chosen in place of the correct Hamiltonian H, where $||H - H_e||_1 \leq \delta$. Note here, $||A||_1 = tr\sqrt{A^\dagger A}$ represents the *trace norm* of a matrix A. As a result, the error occurring in the weak values are given by,

$$\begin{aligned} \Delta_k &= |\langle H\rangle^{(k)}_{\rho\otimes\rho} - \langle H_e\rangle^{(k)}_{\rho\otimes\rho}| = \frac{|\langle u_k|\,\rho\otimes\rho(H-H_e)\,|u_k\rangle\,|}{\langle u_k|\,\rho\otimes\rho\,|u_k\rangle} \\ &\leq \frac{|\langle u_k|\,\rho\otimes\rho(H-H_e)\,|u_k\rangle\,|}{m} \end{aligned} \tag{5.17}$$

where m is the minimum of $\big\{\{p,q,r,s\}\times\{p,q,r,s\}\big\}$ [1] and is always positive. Note that, in obtaining the above inequality we used the fact $\langle u_k|\,\rho\otimes\rho\,|u_k\rangle \in \big\{\{p,q,r,s\}\times\{p,q,r,s\}\big\}$ and it can also be seen that in our protocol, the weak value for k^{th} outcome is only measured when $\langle u_k|\,\rho\otimes\rho\,|u_k\rangle \neq 0$. As a result, the denominator never vanishes in Eq. (5.17). Now, consider the eigenvalue decomposition $H - H_e = \sum_i \lambda_i\,|i\rangle\,\langle i|$, where $\{|i\rangle\}_i$ forms an orthonormal basis in $\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_A\otimes\mathcal{H}_B$, and also note that $||H-H_e||_1 = \sum_i|\lambda_i|$. As a result,

$$\begin{aligned} \Delta_k &\leq \frac{|\sum_i \lambda_i\,\langle u_k|\,\rho\otimes\rho\,|i\rangle\,\langle i|u_k\rangle|}{m} \\ &\leq \frac{1}{m}\sum_i |\lambda_i|\;|\,\langle u_k|\,\rho\otimes\rho\,|i\rangle\,\langle i|u_k\rangle|. \end{aligned} \tag{5.18}$$

Since $0 \leq \rho \leq \mathbb{I}$, it can be easily seen that $|\,\langle u_k|\,\rho\otimes\rho\,|i\rangle\,\langle i|u_k\rangle| \leq 1$. Thus we have,

$$\Delta_k \leq \frac{1}{m}\sum_i|\lambda_i| \leq \frac{||H-H_e||_1}{m} \leq \frac{\delta}{m} \tag{5.19}$$

Thus we see, our protocol is robust to errors arising from inappropriate choice of weak interaction.

[1] Here for any set $\{a,b\}$, we define the cartesian product as $\{a,b\}\times\{a,b\} \equiv \{a^2,\ ab,\ ba,\ b^2\}$.

5.5 Summary

In this chapter, we have proposed a universal entanglement detection protocol for two-qubit quantum states. We consider the most general form of density matrix for two-qubit states, and show that it is enough to have just two copies of the given state to identify if it is entangled or not. Our formulation is based on the determinant based separability criterion and the idea of weak values. Previously in [116], it was demonstrated that one can universally detect entanglement in two-qubit systems using four copies of the state. Our protocol therefore, leads to a clear advantage in terms of resource, as in our case it is sufficient to have just two copies of the state. Our protocol requires only a single projective measurement setting in the computational basis for the purpose of post selection in weak measurement. It is interesting to note that in our protocol the number of copies required for entanglement detection may be further reduced if some partial information about the state is known. Moreover, we have also shown that the procedure of identification is achievable by local operations, if the state in consideration is a pure state. Further, we have shown that the protocol is robust against error arising during application of the weak interaction.

Before concluding, it may be noted that though our scheme reduces the number of measurement settings compared to the universal entanglement witnessing scheme of [116] that requires four copies of the state at a time, this advantage comes at the expense of joint unitary actions on two copies of the state (for arbitrary mixed states). Further work involving quantitative comparison of resources used in our scheme and that employed in other schemes such as in [116] would be needed to obtain a clear idea of practical viability. In this context, one may need to compare the energy cost of creating correlations [144] with the energy cost of doing measurements [145,146] used in the various protocols. Finally, we note that if a similar determinant based criterion for identification of certain class of states is available for higher dimensions, we expect a similar detection protocol such as ours to work therein.

As of now, we have discussed the identification of quantum states for a given scenario. In Chapter 4, we considered 1-sided device independent scenario, in light of which a protocol has been presented to self-test any two-qubit pure entangled state and the corresponding measurement settings. In the present chapter, we show that it is possible to detect if any two-qubit state is entangled or not just by using two copies of the state at a given time, without having any prior knowledge about the state. Now in the next chapter, we discuss about another quantum resource which differs from the previously discussed non-local multipartite quantum correlations on a basic level. This is named as quantum coherence and it does not need to involve more than one parties. Coherence gives the idea of quantumness for any particle and it encapsulates the defining feature of quantum information theory based on the superposition principle. In the next chapter we try to explore the bridge between this physical resource and quantum entanglement. Here we study the interplay of these quantities through some coherent and incoherent operations and see the trade-off between them while entanglement gets generated at the cost of coherence in the input end.

Chapter 6

Coherence and entanglement under three-qubit cloning operations

Coherence and entanglement are the two most crucial resources for various quantum information processing tasks. Here, we study the interplay of coherence and entanglement under the action of different three-qubit quantum cloning operations. Considering certain well-known quantum cloning machines (input state independent and dependent), we provide examples of coherent and incoherent operations performed by them. We show that both the output entanglement and coherence could vanish under incoherent cloning operations. Coherent cloning operations, on the other hand, could be used to construct a universal and optimal coherence machine. It is also shown that under coherent cloning operations, the output two-qubit entanglement could be maximal even if the input coherence is negligible. Also it is possible to generate a fixed amount of entanglement independent of the nature of the input state.

6.1 Introduction and motivation

Out of three non-local quantum correlations (such as enanglement [6], steering [7, 10, 11] and Bell-nonlocality [8]), entanglement is the most widely applied resource in the field of quantum information [45]. Various manifestations of entanglement among discrete, continuous and hybrid physical variables have been studied in the context of applications in information theoretic protocols such as dense coding [2], teleportation [1] and cryptography [3]. Investigations of the resource theory of entanglement have uncovered rich tenets [9], and some surprising features such as intra-particle entanglement [147]. The connection of entanglement with other defining features of quantum theory such as the uncertainty principle has been rigorously examined [148, 149].

Recently, quantum coherence [150] has come to be appreciated as one of the fundamental features of quantum theory. It has been realized that coherence embodies basic quantumness responsible for superposition of quantum states, from which all quantum correlations arise in composite systems. As with entanglement, several measures have been suggested to quantify coherence [151–153]. Interesting connections of coherence with thermodynamic properties of multipartite systems have been pointed out [154–156]. Efforts are on to develop resource theories of coherence enabling it to be used for detection of genuine non-classicality in physical states, and advantage in physical tasks over those performed using classical resources [157–159].

The relation of coherence with other resources in quantum theory forms an interesting arena of study. In a recent work, Streltsov *et al.* [22] have provided an important insight into the linkage of coherence with entanglement. Based upon the observation that two-qubit incoherent operations can generate entanglement only when the input state is coherent, they have shown that the input state coherence provides an upper bound on the generated two-qubit entanglement. In another recent work, the complementarity of local coherence measures has been used to derive a nonlocal advantage of coherence in the form of enabling quantum steering [160]. In entanglement theory, it is known that the robustness (robustness of the state means here that the state does lose less quantum information in the quantum teleportation through noisy channels.) of GHZ and W states depends on the types of noisy channel [161] while W state is more robust against qubit loss [162]. In the resource theory of coherence, Y-Luo et. al. [163] have shown that if one qubit is lost from GHZ state then the state will become incoherent but in case of W state, if one qubit is lost then the remaining two-qubit state remain coherent. Moreover, they have defined inequivalent classes of multipartite coherence states in the same spirit as in entanglement theory. The connection between coherence and nonlocal resources such as entanglement is important to understand from both the perspective of quantum foundations and information theoretic applications, and thus deserves further study in various contexts.

In the present work we pose the question as to how the linkage between coherence and entanglement fares in the presence of additional parties or qubits. Specifically, we study the relationship between two-qubit entanglement and coherence under three-qubit operations. Quantum cloning provides a prototypical example of three-qubit operations, and here we employ coherent and incoherent cloning operations to investigate the connection between coherence and entanglement of the input and output states. For this purpose we consider different categories of cloning machines, such as the Wootters-Zurek [37] mechanism which acts as an incoherent operation, the Buzek-Hillary state independent cloning machine [38] which performs coherent operations. Also we consider phase covariant [39] and state dependent universal [40] cloning machines in order to undertake our study. Cloning could play an efficient role in resource replication, and in the present context we propose an optimal quantum coherence machine using our analysis.

6.2 Entanglement and coherence in reduced two qubit system under incoherent quantum operations

In this section we consider a three qubit incoherent quantum operation and investigate the coherence and entanglement generated in two qubit reduced state when third ancilla qubit is traced out. Coherence is an elementary property of quantum theory, which is basically a measure of quantumness arising from the superposition principle of quantum mechanics. As mentioned in Sec. (2.3), the quantification of coherence is basis dependent and also it might exist in a single-partite systems. In the resource theory of coherence, a state ρ is said to be incoherent if the density matrix of the state does not have non-zero off-diagonal element when written in a given basis, as expressed in Eq. (2.10). Otherwise, it is said to be a coherent state. This definition holds not only for single qubit systems but also for higher dimensional quantum systems. There are different types of measures to quantify the amount of coherence in a given quantum state as mentioned in Sec. (2.3). In our present analysis, we will employ the l_1 norm measure [150] defined as the algebric sum of the off-diagonal elements of the density matrix corresponding to the quantum state in consideration. The mathematical expression is given in Eq. (2.12).

On the other hand, for the purpose of measuring entanglement, here we consider concurrence [164] of the quantum state (as it is sufficient for two qubit scenario) as defined in Eq. (2.2) of Chapter (2). In order to motivate our study, let us here briefly return to the case of two qubit incoherent operations discussed earlier by Streltsov *et al.* [22]. Consider the tensor product of an input coherent state

$$|\psi\rangle_a = c_1 |0\rangle_a + c_2 |1\rangle_a, \qquad |c_1|^2 + |c_2|^2 = 1 \tag{6.1}$$

with the ancilla state $|0\rangle_b$, and the state of the composite system is given by

$$\begin{aligned} |\Phi\rangle_{ab} &= |\psi\rangle_a \otimes |0\rangle_b \\ &= c_1 |00\rangle_{ab} + c_2 |10\rangle_{ab} \end{aligned} \tag{6.2}$$

Now, a two qubit unitary CNOT operation which is given as,

$$U_{CNOT} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| \tag{6.3}$$

applied on $|\Phi\rangle_{ab}$, results in the two qubit state given by

$$|\phi\rangle_{out} = c_1 |00\rangle_{ab} + c_2 |11\rangle_{ab} \tag{6.4}$$

One may now consider the following cases. If either $c_1 = 0$ or $c_2 = 0$, the input state (6.1) is incoherent and it remains an incoherent state even after the application of CNOT operation. Since the CNOT operation takes an incoherent state to another incoherent state and takes the set of incoherent basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ into another $\{|00\rangle, |01\rangle, |11\rangle, |10\rangle\}$, it can be regarded as an incoherent operation. If both $c_1 \neq 0$ and $c_2 \neq 0$, the input state is a coherent state and the application of the CNOT operation on the tensor product (6.2) will generate an entangled state which basically reproduces the fact that to generate entanglement through an incoherent operation one has to start with a coherent state. In this case we find that the amount of entanglement generated is equal to the amount of coherence present in the input state. In general, it has been shown [22] that the maximum entanglement generated by an incoherent operation is given by the amount of coherence present in the input qubit. In other words, a two-qubit incoherent operation generates entanglement, only if the input state has non-vanishing coherence. It is hence natural to ask the question if such a result can be extended to systems involving additional qubits. Note also, that the coherence of the two qubit input state $|\Phi\rangle_{ab}$ is equal to the coherence of the two qubit output state (6.4) and it is given by $2|c_1||c_2|$. Thus, the coherence of the output state depends on the input state parameters. If, on the other hand, we trace out the second system, i.e., the mode b from the two qubit system (6.4), the qubit in mode a is left in an incoherent state. By generating entanglement through this incoherent operation one has to pay the price in terms of reducing the amount of coherence in the outputs ρ_A and ρ_B compared to that present in the input state [22]. In fact, in this case the single-qubit state at the output end is incoherent while we have started with a coherent state. In our subsequent analysis with three qubit operations, we investigate further this issue of the amount of coherence retained in the output states and its relation to the entanglement generated using three qubit operations.

Let us first consider the Wootters-Zurek cloning operation, which is a three qubit quantum operation expressed as [37]

$$|0\rangle |0\rangle |0\rangle \rightarrow |0\rangle |0\rangle |0\rangle \tag{6.5}$$

$$|1\rangle |0\rangle |0\rangle \rightarrow |1\rangle |1\rangle |1\rangle \tag{6.6}$$

where the first ket vector represents the input state, the second ket vector represents the blank state in which the input state is to be copied and the third ket vector represents the machine state. It is clear from equations (6.5) and (6.6) that the cloning operation transforms incoherent input state into incoherent output state and also it takes the set of incoherent basis $\{|0\rangle, |1\rangle\}$ into another $\{|0\rangle, |1\rangle\}$, thus the above defined cloning operation is an example of a three qubit incoherent operation. Note that, in [165] the WZ cloning machine has been studied for higher dimensional systems. From the transformation rule of this type of higher dimensional cloning machines, it is clear that it keeps an incoherent input state incoherent.

Now, if we take the input qubit to be coherent in nature, we see that this incoherent operation (6.5-6.6) does not generate entanglement between the input qubit and the blank qubit, when the ancillary machine state is traced out. Let us take the input qubit to be of the form

$$|\psi^{in}\rangle = \alpha |0\rangle + \beta |1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{6.7}$$

When the state (6.7) passes through the cloning transformation given in (6.5-6.6), the resulting two qubit state at the output end after tracing out the machine qubit is given by

$$\rho_{12}^{out} = |\alpha|^2 |00\rangle \langle 00| + |\beta|^2 |11\rangle \langle 11| \tag{6.8}$$

Also, the density operators of the copy qubits are given by

$$\rho_1^{out} = \rho_2^{out} = |\alpha|^2 |0\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1| \tag{6.9}$$

The following observations can be made from the equations (6.8-6.9). It can easily be seen that the state described by the density operator ρ_{12}^{out} is not entangled. Therefore, the transformation (6.5-6.6) is an example of a three qubit incoherent operation that does not generate entanglement between the input qubit and the blank qubit when machine qubit is traced out, even if we start with a coherent input state. We further find that the state described by the density operator ρ_{12}^{out} is an incoherent two qubit state. The copy qubits generated at the output described by the density operators $\rho_1^{out} = \rho_2^{out}$ are incoherent states too. Now, the quality of copying of the cloning machine can be expressed in terms of the distance between the initial and the reduced copied state at the output end, measured by the Hilbert-Schmidt norm given as,

$$D_a = Tr[(\rho^{in} - \rho_1^{out})^2] \tag{6.10}$$

In case of Wootters-Zurek cloning machine, one obtains the distance as,

$$D_a = 2|\alpha|^2(1 - |\alpha|^2) \tag{6.11}$$

where, D_a is known as the copy quality index. Averaging over all input states, one can obtain the copy quality as,

$$D_a = \frac{1}{3} \tag{6.12}$$

Note that, there is another cloning operation which does not generate entangelement at the output end, named phase covariant cloning [39] which can be regarded as an incoherent operation in single-qubit level (but generates coherence in two-qubit output) and it is given as,

$$\begin{aligned} |0\rangle\,|\Sigma\rangle\,|Q\rangle &\rightarrow [(\frac{1}{2}+\sqrt{\frac{1}{8}})\,|00\rangle + (\frac{1}{2}-\sqrt{\frac{1}{8}})\,|11\rangle]\,|\uparrow\rangle \\ &+\frac{1}{\sqrt{8}}(|01\rangle + |10\rangle)\,|\downarrow\rangle \qquad (6.13) \\ |1\rangle\,|\Sigma\rangle\,|Q\rangle &\rightarrow [(\frac{1}{2}+\sqrt{\frac{1}{8}})\,|11\rangle + (\frac{1}{2}-\sqrt{\frac{1}{8}})\,|00\rangle]\,|\downarrow\rangle \\ &+\frac{1}{\sqrt{8}}(|01\rangle + |10\rangle)\,|\uparrow\rangle \qquad (6.14) \end{aligned}$$

Here, without any loss of generality we consider, $|\Sigma\rangle = |0\rangle$, $|Q\rangle = |0\rangle$, $|\uparrow\rangle = |0\rangle$ and $|\downarrow\rangle = |1\rangle$. Likewise in this scenario, it is never possible to generate any entanglement starting even from a coherent state, as one can check starting from a most general form of single qubit coherent state in computational basis, given in Eq. (6.7) with $\alpha \neq 0$ and $\beta \neq 0$. So this type of cloning machine is also not effective for the purpose of generating entanglement. The above results motivate us to consider next three qubit quantum operations which may not be incoherent and can not only generate entanglement between the input qubit and the blank qubit when ancillary machine qubit is traced out, but also generate coherence in the copy qubits at the output.

6.3 Optimal Universal Two-qubit Quantum Coherence Machine

In this section we will consider the Buzek-Hillary (B-H) cloning operations [38] to see that there exist two classes of three qubit coherent quantum operations that generate coherence in the reduced two qubit system. In the first class, the generated coherence depends on input state parameters, while in the second class, the coherence in reduced two qubit system does not depend on the input state parameters.

To begin with, let us consider a three qubit quantum operation [38] and recall the mathematical expression from Eqs. (2.13,2.14) of Sec. (2.4),

$$
\begin{aligned}
|0\rangle_a |0\rangle_b |0\rangle_c \quad &\rightarrow \quad \sqrt{\frac{2}{3}} |0\rangle_a |0\rangle_b |0\rangle_c + \\
&\quad \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) |1\rangle_c
\end{aligned} \tag{6.15}
$$

$$
\begin{aligned}
|1\rangle_a |0\rangle_b |0\rangle_c \quad &\rightarrow \quad \sqrt{\frac{2}{3}} |1\rangle_a |1\rangle_b |1\rangle_c + \\
&\quad \sqrt{\frac{1}{6}} (|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b) |0\rangle_c
\end{aligned} \tag{6.16}
$$

The above transformation is a two-qubit coherent quantum operation as it takes an incoherent state to two-qubit coherent state. The above transformation is also known as the optimal state independent BH cloning transformation in the $\{|0\rangle, |1\rangle\}$ basis. If we take the partial trace over the ancillary machine qubit c at the output end of (6.15) and (6.16), the corresponding reduced two qubit density operators are given by

$$
\begin{aligned}
\rho_{ab}^{out1} \quad &= \quad \frac{2}{3} |00\rangle \langle 00| + \frac{1}{6} (|01\rangle \langle 01| + |10\rangle \langle 01| \\
&\quad + |01\rangle \langle 10| + |10\rangle \langle 10|)
\end{aligned} \tag{6.17}
$$

$$
\begin{aligned}
\rho_{ab}^{out2} \quad &= \quad \frac{2}{3} |11\rangle \langle 11| + \frac{1}{6} (|01\rangle \langle 01| + |10\rangle \langle 01| \\
&\quad + |01\rangle \langle 10| + |10\rangle \langle 10|)
\end{aligned} \tag{6.18}
$$

The entanglement [25, 166] and coherence of the states ρ_{ab}^{out1} and ρ_{ab}^{out2} are equal and given by $\frac{1}{3}$. Thus, the B-H cloning machine generates a two qubit coherent state starting from an incoherent input qubit.

Let us next consider the input state $|\psi^{in}\rangle$ (6.7) with non-zero state parameters α and β. If we apply the optimal universal B-H cloning transformations given in (6.15) and (6.16) on $|\psi^{in}\rangle$, the two qubit cloned state at the output end comes out to be of the form,

$$
\begin{aligned}
\rho_{ab}^{out} &= \frac{2}{3}|\alpha|^2 |00\rangle\langle 00| + \frac{\sqrt{2}\alpha\beta^*}{3}|00\rangle\langle +| + \frac{\sqrt{2}\alpha^*\beta}{3}|+\rangle\langle 00| + \frac{1}{3}|+\rangle\langle +| \\
&+ \frac{\sqrt{2}\alpha\beta^*}{3}|+\rangle\langle 11| + \frac{\sqrt{2}\alpha^*\beta}{3}|11\rangle\langle +| + \frac{2}{3}|\beta|^2 |11\rangle\langle 11|) \qquad (6.19)
\end{aligned}
$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The amount of coherence contained in the state described by the density operator ρ_{ab}^{out} is given by $\frac{4(\alpha^*\beta+\alpha\beta^*)+1}{3}$. ($*$ denotes the corresponding complex conjugate) Thus, in case of the B-H quantum cloning machine, the generated cloned two qubit output is always coherent. It can be observed that the coherence of the state ρ_{ab}^{out} depends on the state parameters α and β of the input. But, it should be noted that the concurrence of this two party output state turns out to be $\frac{1}{3}$, which is independent of the input state parameters. Also, it is quite interesting to note that this cloning machine generates a constant amount of entanglement starting from any single qubit input state. Hence it can be used as a source of constant entanglement.

We have seen that if we use this coherent quantum operation (6.15-6.16), the coherence of the reduced two qubit output state depends on the input state. It would be interesting to design a universal coherence transformation that transforms an arbitrary state $|\Psi\rangle_{ab}$ which may or may not be coherent, to a two qubit coherent state. We demand the transformation to be universal in the sense that the coherence of the two qubit output state should not be depending on the input state parameters. To construct such a coherence transformation, let us start with the most general form B-H quantum cloning transformation given by

$$
\begin{aligned}
|0\rangle_a |0\rangle_b |Q\rangle_c &\rightarrow |0\rangle_a |0\rangle_b |Q_0\rangle_c + (|0\rangle_a |1\rangle_b + \\
&\quad |1\rangle_a |0\rangle_b) |Y_0\rangle_c \qquad (6.20)
\end{aligned}
$$

$$
\begin{aligned}
|1\rangle_a |0\rangle_b |Q\rangle_c &\rightarrow |1\rangle_a |1\rangle_b |Q_1\rangle_c + (|0\rangle_a |1\rangle_b + \\
&\quad |1\rangle_a |0\rangle_b) |Y_1\rangle_c \qquad (6.21)
\end{aligned}
$$

Unitarity of the transformation gives the relations

$$ {}_c\langle Q_i|Q_i\rangle_c + 2{}_c\langle Y_i|Y_i\rangle_c = 1, \qquad i = 0,1 \tag{6.22} $$

$$ {}_c\langle Y_0|Y_1\rangle_c = 0, \tag{6.23} $$

Let us further assume the following orthogonality relations between the machine state vectors:

$$ {}_c\langle Q_i|Y_i\rangle_c = 0, \quad i = 0,1 \tag{6.24} $$

$$ {}_c\langle Q_0|Q_1\rangle_c = 0, \tag{6.25} $$

First let us apply the cloning transformation given in (6.20) and (6.21) on an incoherent input state, say, $|0\rangle$ ($|1\rangle$). At the output end, the coherence and concurrence of the final two party state (while the state of the ancillary system is traced out) turn out to be the same and it is given by, 2μ, where, μ is given by,

$$ {}_c\langle Y_0|Y_0\rangle_c =_c \langle Y_1|Y_1\rangle_c = \mu \tag{6.26} $$

Secondly, applying the cloning transformation (6.20) and (6.21) on $|\psi^{in}\rangle$ given by Eq.(6.7), and taking the partial trace over the ancillary machine qubit c, we obtain the cloned two qubit state described by the density operator

$$ \begin{aligned} \varrho_{ab}^{out} \;=\; & |\alpha|^2(1-2\mu)|00\rangle\langle 00| + \alpha\beta^*\frac{\nu}{\sqrt{2}}|00\rangle\langle +| + \alpha^*\beta\frac{\nu}{\sqrt{2}}|+\rangle\langle 00| \\ & +2\mu|+\rangle\langle +| + \alpha\beta^*\frac{\nu}{\sqrt{2}}|+\rangle\langle 11| + \alpha^*\beta\frac{\nu}{\sqrt{2}}|11\rangle\langle +| \\ & +|\beta|^2(1-2\mu)|11\rangle\langle 11|) \end{aligned} \tag{6.27} $$

where μ is given as Eq.(6.26) and ν is given by

$$ {}_c\langle Y_0|Q_1\rangle_c =_c \langle Q_0|Y_1\rangle_c = \frac{\nu}{2} \tag{6.28} $$

Using the Schwarz inequality, the range of the parameters μ, ν are given by

$$0 \leq \mu \leq \frac{1}{2} \quad \text{and} \quad 0 \leq \nu \leq 2\sqrt{\mu}\sqrt{1-2\mu} \leq \frac{1}{\sqrt{2}} \tag{6.29}$$

Now, the coherence of the state described by the density operator ϱ_{ab}^{out} is given by

$$C_{l_1}(\varrho_{ab}^{out}) = 2\mu + 2(\alpha^*\beta + \alpha\beta^*)\nu \tag{6.30}$$

The quantity $C_{l_1}(\varrho_{ab}^{out})$ is input state independent if $\nu = 0$. In this case Eq.(6.30) reduces to $C_{l_1}(\varrho_{ab}^{out}) = 2\mu$. The maximum value of $C_{l_1}(\varrho_{ab}^{out})$ can be obtained by putting $\mu = \frac{1}{2}$, which leads to

$$C_{l_1}(\varrho_{ab}^{out}) = 1 \tag{6.31}$$

For these particular values of μ and ν, it can be seen that the concurrence of the two party state is maximum. Also note that, the copy quality index in this case turns out to be $\frac{1}{18}$ which is much less than that of WZ cloning machine (hence better quality of cloning) and also independent of the input state parameter [38]. Eq. (6.31) is the evidence of the fact that the coherence present in the two qubit output state is optimal and independent of the input state parameters. Thus, we are successful in constructing a universal quantum coherence machine starting from the B-H quantum cloning machine. Particularly, the optimal universal quantum coherence transformation is given by

$$|0\rangle_a|0\rangle_b|0\rangle_c \rightarrow \sqrt{\frac{1}{2}}(|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b)|0\rangle_c \tag{6.32}$$

$$|1\rangle_a|0\rangle_b|0\rangle_c \rightarrow \sqrt{\frac{1}{2}}(|0\rangle_a|1\rangle_b + |1\rangle_a|0\rangle_b)|1\rangle_c \tag{6.33}$$

It is now clear that optimal universal quantum coherence transformation can be obtained from B-H quantum cloning transformation by choosing the machine vector in such a way that $\mu = \frac{1}{2}$ and $\nu = 0$. Let us now ascertain how well the

B-H copying machine with machine parameters $\mu = \frac{1}{2}$ and $\nu = 0$ copies the input qubit described by the density operator

$$\begin{aligned} \rho^{in} &= |\psi_{in}\rangle \langle\psi_{in}| \\ &= |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \alpha^*\beta |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1| \end{aligned} \tag{6.34}$$

Since the B-H quantum cloning machine is considered to be symmetric in nature, the copies of the input state at the output end of the copying machine are identical and are given by

$$\begin{aligned} \rho_a^{out} = \rho_b^{out} &= (|\alpha|^2(1-2\mu) + 2\mu)|0\rangle\langle 0| + \sqrt{2}\nu\alpha\beta^*|0\rangle\langle 1| \\ &\quad + \sqrt{2}\nu\alpha^*\beta|1\rangle\langle 0| + (|\beta|^2(1-2\mu) + 2\mu)|1\rangle\langle 1| \end{aligned} \tag{6.35}$$

The distance between ρ^{in} and $\rho_a^{out}/\rho_b^{out}$ can be measured by the Hilbert Schmidt norm, given as

$$\begin{aligned} D_a &= Tr[(\rho^{in} - \rho_a^{out})^2] = Tr[(\rho^{in} - \rho_b^{out})^2] \\ &= 2\mu^2(1 - 4|\alpha|^2|\beta|^2) + 2|\alpha|^2|\beta|^2(\nu - 1)^2 \end{aligned} \tag{6.36}$$

Note that, for $\nu = 1 - 2\mu$, the quality of copy for the B-H cloning machine becomes input state independent and particularly for $\mu = \frac{1}{2}$ and $\nu = 0$, the distance D_a reduces to

$$D_a = \frac{1}{2} \tag{6.37}$$

Equation (6.37) indicates that the quality of the copy also does not depend on the input state parameter. Therefore, for the cloning machine parameter $\mu = \frac{1}{2}$ and $\nu = 0$, the B-H quantum cloning machine becomes an input state independent quantum cloning machine, but it should be noted that this cloning machine is not optimal in terms of quality of cloning.

6.4 Generation of entanglement from coherent operations

A quantum operation is said to be a coherent operation if it generates coherence even from an incoherent state. It has been already seen that under the application of any coherent operation it is possible to generate entanglement starting even from an incoherent state. Unlike incoherent quantum operations (free operation in coherence resource theory), we have seen that when a coherent operation (6.15) acts on the tensor product of an incoherent input state and an incoherent ancilla state, it generates entanglement in the two qubit reduced state when the ancilla state is traced out. We found that the amount of entanglement generated in the two qubit reduced state is equal to the amount of coherence in it. This leads us to the following result.

Result: If we construct a coherent operation Λ^c in such a way that it generates a two qubit mixed state of the form

$$\begin{aligned}\rho_{AB} &= a|00\rangle\langle 00| + b|01\rangle\langle 01| + c|01\rangle\langle 10| + c^*|10\rangle\langle 01| \\ &\quad + d|10\rangle\langle 10| + e|11\rangle\langle 11| \end{aligned} \tag{6.38}$$

when applied on the tensor product of an incoherent input state and an incoherent ancilla state, the output entanglement and coherence are related by

$$C(\rho_{AB}) \leq C_{l_1}(\rho_{AB}) \tag{6.39}$$

where $C(\rho_{AB})$ is the concurrence of the output two qubit state and C_{l_1} is the l_1 norm measure of coherence of the corresponding state.

Proof: It is known that the concurrence of the two qubit mixed entangled state (6.38) is given by [167, 168],

$$C(\rho_{AB}) = max\{0, 2(|c| - \sqrt{ae})\} \tag{6.40}$$

From (6.40), we find that

$$C(\rho_{AB}) \leq 2|c| \tag{6.41}$$

Again, the l_1 norm of coherence of the two qubit mixed state (6.38) is given by

$$C_{l_1}(\rho_{AB}) = |c| + |c| = 2|c| \tag{6.42}$$

Using (6.41) and (6.42), we have

$$C(\rho_{AB}) \leq C_{l_1}(\rho_{AB}) \tag{6.43}$$

Hence proved.

In this section, we consider a coherent operation in the form of B-H quantum cloning machine to study the entanglement structure of the two qubit output state. Depending on the (coherent/incoherent) nature of the input state, we analyze the entanglement structure of two qubit state at the output end of the cloning machine. First, let us consider the case when the input state to be cloned is an incoherent state which is either of the form $|0\rangle$ or $|1\rangle$. When $|0\rangle$ goes through the cloning transformation given by Eqs.(6.20) and (6.21), the two qubit output state is given by

$$\rho_{ab}^{out3} \quad = \quad (1-2\mu)|00\rangle\langle 00| + 2\mu|+\rangle\langle +| \tag{6.44}$$

It is clear that the concurrence of the two qubit state ρ_{ab}^{out3} given by Eq.(6.44) is non-zero and given by 2μ. A similar result can be obtained when the input state to be cloned is of the form $|1\rangle$. Therefore, the general B-H quantum cloning transformation generates an entangled two qubit cloned state when the input state is an incoherent. A maximally entangled state is generated when $\mu = \frac{1}{2}$. The structure of the cloning transformation that generates the maximally entangled state of two cloned copies out of the incoherent input state is the same as the state independent quantum coherence transformation given by (6.32-6.33).

In the second scenario, let us consider that the input state to be cloned is a coherent state $|\psi_{in}\rangle$ given by (6.7). When the general B-H quantum cloning transformation is applied on $|\psi_{in}\rangle$ and tracing out the cloning machine state vector, the resulting two qubit state of two cloned copies is entangled. We study the entanglement of the output two qubit state for two cases.

Case-I: If we perform a state independent B-H quantum cloning transformation given by (6.20) and (6.21) with $\nu = 1 - 2\mu$, on any arbitrary coherent input state $|\psi_{in}\rangle$, the output state is given by

$$\begin{aligned}\varrho_{ab}^{out} &= |\alpha|^2(1-2\mu)|00\rangle\langle 00| + \alpha\beta^*\frac{1-2\mu}{\sqrt{2}}|00\rangle\langle +| + \alpha^*\beta\frac{1-2\mu}{\sqrt{2}}|+\rangle\langle 00| \\ &\quad +2\mu|+\rangle\langle +| + \alpha\beta^*\frac{1-2\mu}{\sqrt{2}}|+\rangle\langle 11| + \alpha^*\beta\frac{1-2\mu}{\sqrt{2}}|11\rangle\langle +| \\ &\quad +|\beta|^2(1-2\mu)|11\rangle\langle 11|) \end{aligned} \tag{6.45}$$

We find that the generated two qubit cloned state is entangled and it is clearly evident from the plot given below. From the plot, it can be seen that there exist

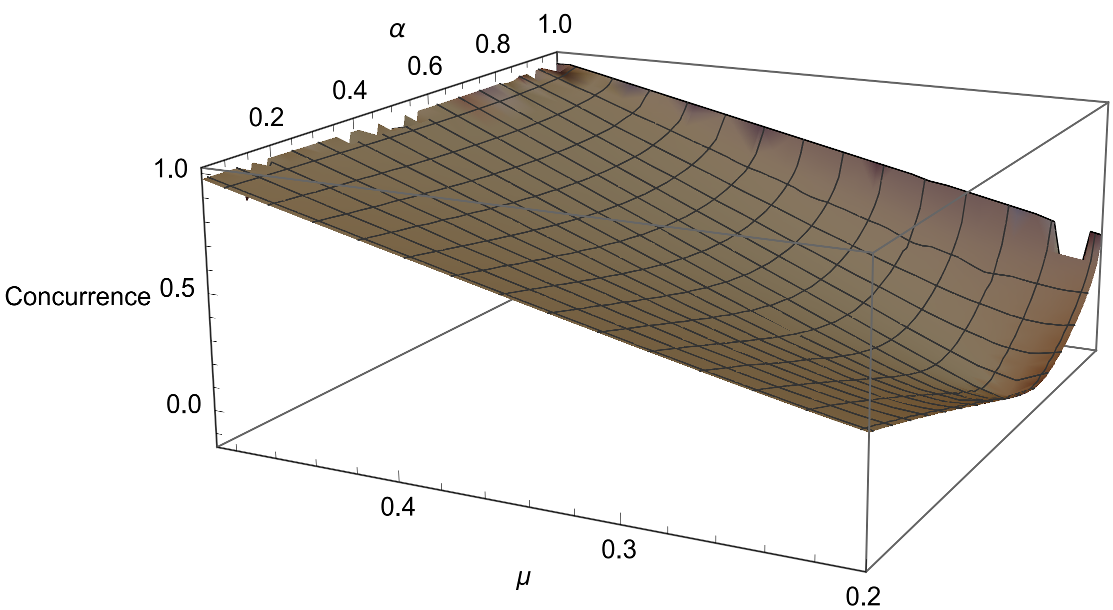

Figure 6.1: Concurrence of the two qubit output state is plotted against the machine parameter μ and the input state parameter α.

state independent B-H quantum cloning transformations that cannot be used to generate two qubit entangled states. Additionally, one may note that the optimal state independent B-H quantum cloning machine can be used to generate a two qubit cloned state from a coherent input state. Also, one may observe that there exists a cloning transformation which generates maximum entanglement at the output end even when the coherence of the input is negligible.

Case-II: If we apply the optimal state independent quantum coherence transformation given by (6.32-6.33) on the coherent input state $|\psi_{in}\rangle$ (or may be on any incoherent input state, i.e. $|\psi_{in}\rangle$ either with $\beta = 0$ or $\alpha = 0$), then the two party output state is given by,

$$\varrho_{ab}^{out} = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) \tag{6.46}$$

The concurrence of this state is unity for any type of input state. So even for negligible amount of coherence it generates maximal entanglement.

6.5 Study of coherence and entanglement in case of general state dependent cloning machine

Now let us consider the cloner that operates as a unitary operation [40] on the composite Hilbert space of the three party state and it is given as,

$$\begin{aligned} |0\rangle|0\rangle|X\rangle \;\;\rightarrow\;\; & a|00\rangle|A\rangle + b_1|01\rangle|B_1\rangle + b_2|10\rangle|B_2\rangle + \\ & c|11\rangle|C\rangle \end{aligned} \tag{6.47}$$

$$\begin{aligned} |1\rangle|0\rangle|X\rangle \;\;\rightarrow\;\; & \tilde{a}|11\rangle\left|\tilde{A}\right\rangle + \tilde{b_1}|01\rangle\left|\tilde{B_1}\right\rangle + \tilde{b_2}|10\rangle\left|\tilde{B_2}\right\rangle + \\ & \tilde{c}|00\rangle\left|\tilde{C}\right\rangle \end{aligned} \tag{6.48}$$

The above cloning operations are introduced in Eqs. (2.15, 2.16) of Sec. (2.4) and are corresponding to the incoherent input states $|0\rangle$ and $|1\rangle$ respectively. Here the state $|X\rangle$ represents the initial ancillary machine state and $|A\rangle$, $|B_1\rangle$, $|B_2\rangle$, $|C\rangle$, $\left|\tilde{A}\right\rangle$, $\left|\tilde{B_1}\right\rangle$, $\left|\tilde{B_2}\right\rangle$, $\left|\tilde{C}\right\rangle$ signify the ancillary machine state at the output end. As the operation of cloning is unitary, the coefficients in each case should satisfy the normalization conditions,

$$a^2 + {b_1}^2 + {b_2}^2 + c^2 = 1 \tag{6.49}$$

$$\tilde{a}^2 + \tilde{b_1}^2 + \tilde{b_2}^2 + \tilde{c}^2 = 1 \tag{6.50}$$

Here, we chose that $c = 0$ and $\tilde{c} = 0$ as the terms corresponding to these coefficients do not produce any productive output (neither it copies the state properly nor gives back the original state). Now for our convenience let us choose (without any loss of generality), $|A\rangle = |0\rangle, |B_1\rangle = |1\rangle, |B_2\rangle = |1\rangle$ and hence $\left|\tilde{A}\right\rangle = |1\rangle, \left|\tilde{B_1}\right\rangle = |0\rangle, \left|\tilde{B_2}\right\rangle = |0\rangle$. Let us first start with an incoherent state $|0\rangle$ ($|1\rangle$), and under the transformation 6.47 (6.48) it can be observed that the single party cloned state at the output end is incoherent in nature. So, we can can call it as a single-qubit incoherent cloning operation. It is interesting to notice that under such type of cloning operation on these incoherent states, it is possible to generate an entangled state at the output end, whose concurrence is given by, $2b_1$ (or, $2\tilde{b_1}$) with $b_1 = b_2$ (and, $\tilde{b_1} = \tilde{b_2}$)(assuming the cloning machine to be symmetric). Now let us consider the initial state to be coherent as given in, 6.7. Now after passing through the cloning machine, the output three party state becomes, $|\psi_{out}\rangle = \alpha[a\,|000\rangle + b_1\,|011\rangle + b_2\,|101\rangle] + \beta[\tilde{a}\,|111\rangle + \tilde{b_1}\,|100\rangle + \tilde{b_2}\,|010\rangle]$. Now to find the coherence of the copied state or the original state after the operation, one needs to trace out the two party state with respect to either the first party or with respect to the second party. For this cloning machine to be symmetric in nature, one should have the coherence of both the states to be equal. By assuming the symmetry of the operation one can get the coherence as $2(\tilde{a}b_2 + a\tilde{b_2})\alpha\beta = 2(\tilde{a}b_2 + a\tilde{b_2})\alpha\sqrt{1-\alpha^2}$. Now optimization with respect to $a, \tilde{a}, b_1, \tilde{b_1}$ gives that the maximum value of coherence of the final one party state for this type of cloning is $\sqrt{2}\alpha\beta$. The corresponding values of the parameters are $a = \frac{3}{4\sqrt{2}} = 0.695654$, $b_1 = b_2 = \frac{\sqrt{23}}{8} = 0.507969$, $\tilde{a} = \frac{\sqrt{\frac{23}{2}}}{4} = 0.718377$ and $\tilde{b_1} = \tilde{b_2} = \frac{3}{8} = 0.491902$. One can easily see that compared to the state independent cloning the coherence of the final state is better in this case. Also, the entanglement also shows an improvement with respect to optimal BH cloning machine. The variation of concurrence with respect to the state parameter, for the optimized set of parameters of the machine is shown in Fig. (6.2) and it can be seen for most of the values of state parameter α the output two party state remains entangled.

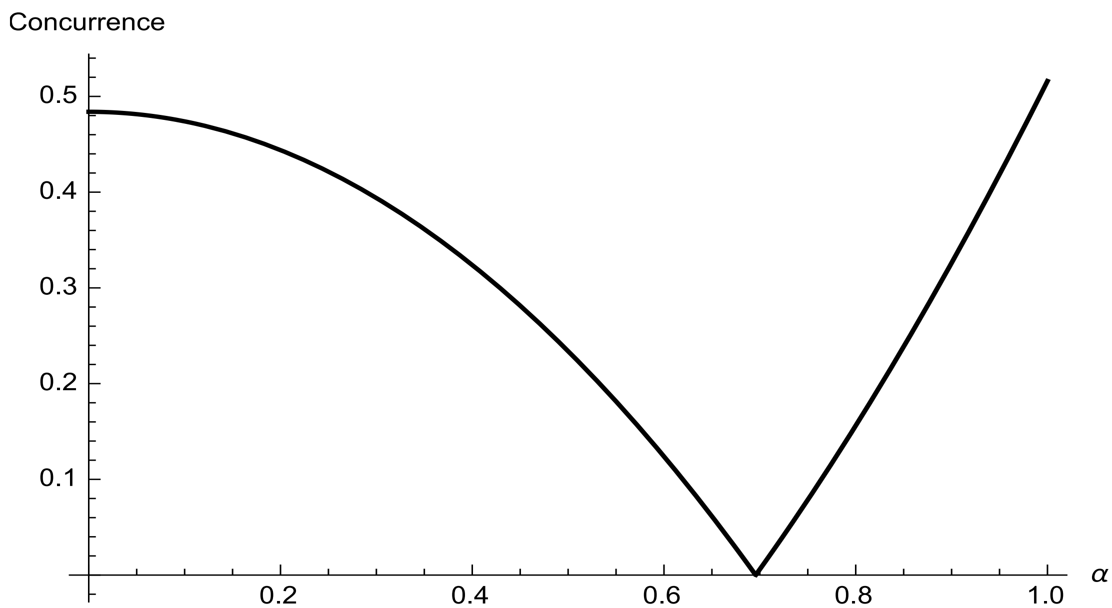

Figure 6.2: Concurrence of the two qubit output state for general state dependent cloning machine is plotted against the input stateparameter α.

6.6 Summary

In this work we have considered three qubit cloning operations and studied the two qubit output coherence and entanglement. The cloning operations have different copy quality indices and here we have considered cloning machines with different copying efficiency. In some cases it is independent of the input state parameters and for others, efficiency is dependent on the same. Recently, a bound has been obtained on the two qubit entanglement in terms of the coherence of a single qubit input state when an incoherent operation is performed on it [22]. Our motivation for the present study is to investigate further the connection between entanglement and coherence in the context of cloning operations involving additional qubits. For this purpose we have considered here two types of well known cloning operations, *viz.*, the Wootters-Zurek copier [37], and the Buzek-Hillery copier [38]. As we have discussed, our cloning operations could be categorized into three qubit coherent and incoherent operations. We have shown that the WZ cloning machine does not generate either coherence or entanglement at the output. Cloning operations may be regarded as resource replicators in quantum information processing. In the present work we next show that the BH copier could act as a universal coherence machine that generates a fixed amount of coherence in the two qubit output state irrespective of the input state parameters. Under the action of coherent cloning operations, a relation is obtained among the two qubit output coherence and entanglement. We have further shown that under such operations, the output entanglement could be maximal even if the input state coherence is negligible.

Chapter 7

Summary and conclusions

The theory of quantum correlations plays a crucial role in quantum information processing tasks, quantum computation and quantum controls. Hence the subject of quantum correlations is a well-studied and vastly explored subject in the literature of quantum information. The motivation of this thesis was to provide some useful tools and results which could make some of these correlations more realizable physically.

In the thesis, we mainly deal with quantum entanglement and quantum steering. Both of these correlations show quite a fragile nature when exposed to the noise of environment. To perform any process in laboratory involves different sort of environmental interactions. As described in Chapter (3), there exists different theoretical noise models which govern the effect of the decoherence caused due to the noise, physically. In this thesis we consider one of them, the generalized amplitude damping channel (GADC) and prescribe a general preservation technique which is based on the structure and the evolution of a given channel. First, from the operator sum representation of the corresponding channel we find the unitary dilation considering suitable ancillary state. This unitary is not unique in nature. After obtaining the unitary, the inverse of the same is calculated. This inverse unitary dilation gives another CPTP map, corresponding to which there exists a Kraus representation giving the inverse effect of the initial channel. We employ these individual Kraus operators as elements of a POVM. This selective implementation of the POVM shows a fruitful improvement of quantum correlations for a broad range of channel parameters even after passing through the GADC. Now as the unitary dilation is not unique, it leaves us with the scope of creating suitable POVM for the particular situation under consideration. In this thesis, we choose a particular channel under which the system is evolving, but the method that has been proposed here, has the potential to deal with different kinds of noise models as the procedure just involves the basic structure of the given noisy channel. As a future direction it would be interesting to employ this model to preserve important resource for other known noise models, and to compare the effect with other noise-reduction or error-correcting procedures. Also, the unitary dilations chosen corresponding to the particular channel are completely arbitrary choices. It would be interesting and essential to find the most suitable unitary dilation required for a particular scenario by optimizing over few other parameters. This optimization can be done numerically by constraining ourselves to the cases when the chosen unitary action does not generate much correlation between the initial system state and the ancillary qubits, hence destroying the minimal correlation within the system itself.

In the next part of the thesis, the discussion is focused on the recognization of a particular quantum state in a given scenario of quantum correlation. In Chapter (4) it has been shown that it is possible to self-test any pure two-qubit entangled state in 1-sided device independent scenario with the help of steering inequalities. The maximal violation of FGI of steering exactly identifies the class of the state as the class of pure states, as the maximal violation is obtained if and only if the state under consideration is a pure state. From the value of the violation of this inequality, the corresponding measurement settings is also obtained. Afterwards, with the help of another steering inequality (ACHSH or CFFW of steering) or by calculating a quantity mutual predictability it is shown to be possible to identify the particular state (up to some local unitary evolution) in hand. Also, it has been shown that there exists an one-sided isometry structure which gives a correspondence from the system in dimension $d \otimes 2$ to that in $2 \otimes 2$. Now, the robustness of the procedure should be studied in terms of the improper choice of measurement settings and non-exact value of the violation of the steering inequalities, as the future directions. Also, it would be interesting to extend the work for multipartite scenario, and to identify different classes of pure entangled state in that case.

Out of all types of quantum correlations, entanglement is the most explored and useful resource in the field of information processing. Separability problems dealing with the identification of any arbitrary quantum state, are the fundamental problem in this direction and it holds a position of great significance for ages. In the Chapter (5) we discuss an entanglement detection problem motivated towards the minimalization of the resource requirement with respect to previous protocols. Here, it has been shown that it is possible to identify if a given unknown state is entangled or not, with the help of two copies of the state supplied at a given time. To restrict ourselves to just two copies of the state, we use the technique of weak measurement, and by obtaining the weak values corresponding to a global Hamiltonian we show that it is possible to exactly find the elements of the density matrics in consideration, while the post-selective measurement is done in the computational basis. At the end of the procedure of detecting the entanglement using two copies of the state, eventually it identifies all the parameters of the density matrix, which is nothing but the complete identification of the state. So, as a future direction, it is important to compare the resource requirement of this procedure with that of a complete state tomography. Also, it would be interesting to find the thermodynamic energy cost for the implementation of the global Hamiltonian that has been used in the procedure.

As mentioned before, coherence and entanglement are the two most crucial resources for various quantum information processing tasks. Here, in Chapter (6) we study the interplay of coherence and entanglement under the action of different three-qubit quantum cloning operations. Considering certain well-known quantum cloning machines (input state independent and dependent), we provide examples of coherent and incoherent operations performed by them. We show that both the output entanglement and coherence could vanish under incoherent cloning operations. Coherent cloning operations, on the other hand, could be used to construct a universal and optimal coherence machine. It is also shown that under coherent cloning operations, the output two-qubit entanglement could be maximal even if the input coherence is negligible.

Overall, this thesis explores the ideas regarding different non-local quantum correlations. It concerns itself to make these correlations to be more resourceful for different physical tasks, in presence of noisy environment. On the other hand, it explores the detection mechanisms of them as well as interconversion of various resources of quantum information theory.

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